Advanced Mathematics

for Rwandan Schools

Student's Book

Senior Six

FOREWORD

Dear Student,

Rwanda Basic Education Board (REB) is honored to present senior six Mathematics book for students of advanced level where Mathematics is a major subject. This book will serve as a guide to competence-based teaching and learning to ensure consistency and coherence in the learning of the Mathematics. The Rwandan educational philosophy is to ensure that you achieve full potential at every level of education which will prepare you to be well integrated in society and exploit employment opportunities.

The government of Rwanda emphasizes the importance of aligning teaching and learning materials with the syllabus to facilitate your learning process. Many factors influence what you learn, how well you learn and the competences you acquire. Those factors include the relevance of the specific content, the quality of teachers' pedagogical approaches, the assessment strategies and the instructional materials available. In this book, we paid special attention to the activities that facilitate the learning process in which you can develop your ideas and make new discoveries during concrete activities carried out individually or with peers.

In competence-based curriculum, learning is considered as a process of active building and developing knowledge and meanings by the learner where concepts are mainly introduced by an activity, situation or scenario that helps the learner to construct knowledge, develop skills and acquire positive attitudes and values.

For efficiency use of this textbook, your role is to:

- Work on given activities which lead to the development of skills;
- Share relevant information with other learners through presentations, discussions, group work and other active learning techniques such as role play, case studies, investigation and research in the library, on internet or outside;
- Participate and take responsibility for your own learning;
- Draw conclusions based on the findings from the learning activities.

To facilitate you in doing activities, the content of this book is self explanatory so that you can easily use it yourself, acquire and assess your competences. The book is made of units as presented in the syllabus. Each unit has the following structure: the key unit competence is given and it is followed by the introductory activity before the development of mathematical concepts that are connected to real world problems or to other sciences.

The development of each concept has the following points:

- It starts by a learning activity: it is a hand on well set activity to be done by students in order to generate the concept to be learnt;
- Main elements of the content to be emphasized;
- Worked examples; and
- Application activities which are activities to be done by the user to consolidate competences or to assess the achievement of objectives.

Even though the book has some worked examples, you will succeed on the application activities depending on your ways of reading, questioning, thinking and grappling ideas of calculus not by searching for similar-looking worked out examples.

Furthermore, to succeed in Mathematics, you are asked to keep trying; sometimes you will find concepts that need to be worked at before you completely understand. The only way to really grasp such a concept is to think about it and work-related problems found in other reference books.

I wish to sincerely express my appreciation to the people who contributed towards the editing of this book, particularly, REB staffs and teachers for their technical support.

Any comment or contribution would be welcome to the improvement of this text book for the next edition.

Dr. NDAYAMBAJE Irénée

Director General, REB

ACKNOWLEDGEMENT

I wish to express my appreciation to the people who played a major role in the development and the editing of senior six Mathematics book for students of advanced level where Mathematics is a major subject. It would not have been successful without active participation of different education stakeholders.

I owe gratitude to Curriculum Officers and teachers whose efforts during the editing exercise of this book were very much valuable.

Finally, my word of gratitude goes to the Rwanda Basic Education Board staffs who were involved in the whole process of in-house textbook production.

Joan MURUNGI

Head of Curriculum, Teaching and learning Resources Department

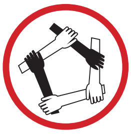
Table of Content

Icons

To guide you, each activity in the book is marked by a symbol or icon to show you what kind of activity it is. The icons are as follows:

Practical Activity icon

The hand indicates a practical activity such as curve sketching, draw figures, to have a selection of objects individually or in a group and then present your results or comments.



Group Work icon

Group work means that you are expected to discuss something in groups and report back on what your group discussed. In this way, you learn from each other, and how to work together as a group to address or solve a problem.

Pairing Activity icon

This means that you are required to do the activity in pairs, exchange ideas and write down your results.

Research Activity icon

Some activities require you to do research either by reading textbooks or using the internet.

Unit 1

Complex Numbers

Introductory activity

Solve in the set of real number the following equations:

1) $x^2+6x+8=0$

2) $x^2+4=0$

Does every quadratic equation have solution in $\mathbb{R}$?

What happens to the equation $x^2+4=0$ if we conventionally accept a number i such that $i^2=-1$? Can now any quadratic equation be solved?

The history of complex numbers goes back to the ancient Greeks who decided that no number existed that satisfies $x^2+1=0$ in $\mathbb{R}$. Many mathematicians contributed to the full development of complex numbers. The rules for addition, subtraction, multiplication, and division of complex numbers were developed by the Italian mathematician **Rafael Bombelli**.

Objectives

By the end of this unit, a student will be able to:

- Identify a real part and imaginary part of a complex number.
- Convert a complex number from one form to another.
- Represent a complex number on Argand diagram.
- State De Moivre's formula and Euler's formulae.
- Apply the properties of complex numbers to perform operations on complex numbers in algebraic form, in polar form or in exponential form.
- Find the modulus and the n^{th} roots of a complex number.
- Solve in the set of complex numbers a linear or quadratic equation.
- Use the properties of complex numbers to factorise a polynomial and to solve a polynomial equation in the set of complex numbers.

- Apply complex numbers in trigonometry and alternating current problems.

1.1. Concepts of complex numbers

Activity 1.1

1. Find two numbers, a and b, whose sum is 6 and product 18.
2. Considering that $\sqrt{-1} = i$, find again the value of a and b.
3. Are a and b elements of $\mathbb{R}$?

From activity 1.1, we see that there are no real solutions since the square root of a negative real number does not exist in set of real numbers, but if you assume that $\sqrt{-1} = i$ you can find the solution. The numbers found in activity 1.1 are called **complex numbers**.

A **complex number** is a number that can be put in the form $a+bi$, where a and b are real numbers and $i = \sqrt{-1}$ (i being the first letter of the word "imaginary").

The set of all complex numbers is denoted by $\mathbb{C}$ and is defined as $\mathbb{C} = \left\{ z = a+bi : a,b \in \mathbb{R} \ \ and \ \ i^2 = -1 \right\}$

The real number a of the complex number $z = a+bi$ is called the **real part** of z and denoted by $Re(z)$ or $\Re(z)$; the real number b is called the **imaginary part** of z and denoted by $Im(z)$ or $\Im(z)$.

Example 1.1

Give two examples of complex numbers.

Solution

There are several answers. For example $-3.5+2i$ and $4-6i$, where $i^2 = -1$, are complex numbers.

Example 1.2

Show the real part and imaginary part of the complex number $-3+4i$.

Solution

$\text{Re}(-3+4i)=-3$ and $\text{Im}(-3+4i)=4$

Remarks

a) It is common to write a for $a+0i$ and bi for $0+bi$.

Moreover, when the imaginary part is negative, it is common to write $a-bi$ with b> 0 instead of $a+(-b)i$, for example $3-4i$ instead of $3+(-4)i$.

b) A complex number whose real part is zero is said to be **purely imaginary** whereas a complex number whose imaginary part is **zero** is said to be a **real number** or simply **real**.

Therefore, all elements of $\mathbb{R}$ are elements of $\mathbb{C}$; and we can simply write $\mathbb{R}\subset\mathbb{C}$.

Notice

- We can write $a+ib$ instead of $a+bi$ (scalar multiplication between b and i is commutative). Also, we can write $bi+a$ instead of $a+bi$ (addition is commutative).
- In some disciplines, in particular electromagnetism and electrical engineering, j is used instead of i, since i is frequently used for electric current. In these cases, complex numbers are written as $a+bj$.
- For comparison operations, only equality of complex numbers is defined. The comparison using < or > are not defined for complex numbers.

Application activity 1.1

1. Show the real and imaginary parts of the following complex numbers:

a) $z=45i$ b) $z=-3$

c) $z=-1+3i$ d) $z=7i-10$

2. For each of the following, say if the complex number is purely imaginary, real or neither.

a) $z = 13$ b) $z = -4i$

c) $z = -7i$ d) $z = 9i - 18$

1.2. Algebraic form of a complex number

Recall that the set of all complex numbers is denoted by $\mathbb{C}$ and is defined as $\mathbb{C} = \{z = a + bi : a, b \in \mathbb{R} \text{ and } i^2 = -1\}$.

$z = a + bi$ is the algebraic (or standard or Cartesian or rectangular) form of the complex number z.

1.2.1. Definition and properties of "i"

Activity 1.2

Using the fact that $i^2 = -1$, find the value of $i^3, i^4, i^5, i^6, i^7, i^8$ and i^9.

Find the general formula of calculating i^k, $k \in \mathbb{N}$.

For a complex number $z = a + bi$, i is called an imaginary unit.

From activity 1.2, we get the important remark:

Properties of imaginary unit i

The powers of imaginary unit are: $i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$.

If we continue, we return to the same results; the imaginary unit, i, "cycles" through 4 different values each time we multiply as it is illustrated in figure 1.1.

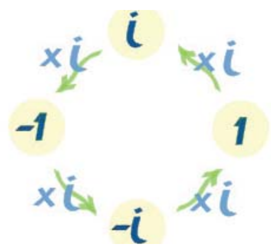


Figure 1.1. *Rotation of imaginary unit i*

Other exponents may be regarded as $4k+m$, $k=0,1,2,3,4,5,...$ and $m=0,1,2,3$.

Thus, the following relations may be used:

$i^{4k}=1,\ i^{4k+1}=i,\ i^{4k+2}=-1,\ i^{4k+3}=-i$

Example 1.3

Find the value of i^{48}, i^{801}, i^{142} and i^{22775}

Solution

$i^{48}=i^{4\times 12}=1,\qquad i^{801}=i^{4\times 200+1}=i,$

$i^{142}=i^{4\times 35+2}=-1,\qquad i^{22775}=i^{4\times 5693+3}=-i$

Application activity 1.2

Find the value of:

1. i^{10}
2. i^{1213}
3. i^{2244}
4. i^{46787}
5. i^{12345}
6. i^{45687}

1.2.2. Geometric representation of complex numbers

Activity 1.3

In xy plane, represent the points $A(1,2)$, $B(-3,2)$, $C(2,-1)$ and $D(-2,-3)$.

A complex number can be visually represented as a pair of numbers (a,b) forming a vector from the origin or point on a diagram called **Argand diagram** (or **Argand plane**), named after Jean-Robert Argand, representing the complex plane. This plane is also called Gauss plane. The $x-axis$ is called the real axis and is denoted by **Re** while the $y-axis$ is known as the imaginary axis; denoted **Im** as illustrated in fig.1.2.

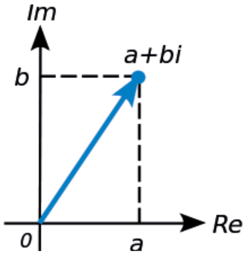


Figure 1.2. *Geometric representation of a complex number*

The Argand diagram fig 1.2 represents complex number $z=a+bi$ both as a point $P(\mathrm{a},\mathrm{b})$ and as a vector $\overrightarrow{OP}$.

$z=(a,b)$ is a **geometric** form of the complex number z .

Notice

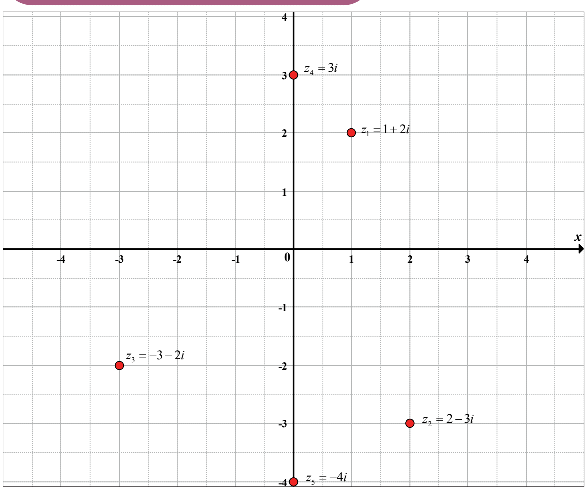
In complex plane, we will no longer talk about coordinates but affixes. The **affix** $z=a+bi$ of a point is plotted as a point and position vector on an Argand diagram; $a+bi$ is the **rectangular expression** of the point.

Example 1.4

Plot in the same Argand diagram the complex numbers

$z_1=1+2i,\ z_2=2-3i,\ z_3=-3-2i,\ z_4=3i$ and $z_5=-4i$.

Solution



Application activity 1.3

Represent on the same Argand diagram the complex numbers:

1) $z_1 = 2-2i$ 2) $z_2 = 3$ 3) $z_3 = -1-i$

4) $z_4 = 4i$ 5) $z_5 = 2+i$ 6) $z_6 = -2-3i$

7) $z_7 = -5i$ 8) $z_8 = 3+3i$

1.2.3. Modulus of a complex number

Activity 1.4

Plot the following complex numbers in the Argand diagram and hence, for each, find its distance from origin.

1. $z = -8$ 2. $z = 2i$ 3. $z = -3+7i$ 4. $z = 3-4i$

The distance from origin to the point (x, y) corresponding to the complex number $z = x + yi$ is called the **modulus** (or **magnitude** or **absolute value**) of z and is denoted by $|z|$ or $|x+iy|$:. Thus, modulus of z is given by $r = |z| = \sqrt{x^2 + y^2}$.

Example 1.5

Find the modulus of $4-3i$

Solution

$$|4-3i| = \sqrt{16+9} = 5$$

Example 1.6

Find the modulus of i

Solution

$$|i| = \sqrt{0^2 + 1^2} = 1$$

Example 1.7

Find the modulus of -3

Solution

$$|-3| = \sqrt{(-3)^2 + 0^2} = 3$$

Example 1.8

Find the modulus of $\frac{1}{2}\left(1+i\sqrt{3}\right)$

Solution

$$\left|\frac{1}{2}\left(1+i\sqrt{3}\right)\right| = \frac{1}{2}\left|1+i\sqrt{3}\right|$$

$$= \frac{1}{2}\sqrt{1+3} = 1$$

Properties of modulus

Let z, w be complex numbers different from 0, thus

a) $|z|^2 = [R_e(z)]^2 + [Im(z)]^2$ b) $|z|^2 = z\bar{z}$

c) $\text{Re}(z) \le |\text{Re}(z)| \le |z|$ d) $\text{Im}(z) \le |\text{Im}(z)| \le |z|$

e) $|zw| = |z||w|$ f) $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$

g) $|z+w| \le |z| + |w|$ i) $|z-w| \ge |z| - |w|$

Example 1.9

Find the modulus of $\frac{5}{3-4i}$

Solution

$$\left|\frac{5}{3-4i}\right| = \frac{|5|}{|3-4i|}$$

$$= \frac{5}{\sqrt{9+16}} = 1$$

Example 1.10

Find the modulus of $\frac{2+i}{1-3i}$

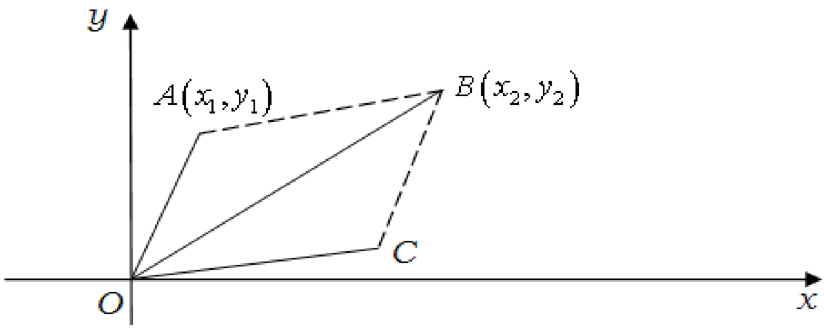
Solution

$$\left|\frac{2+i}{1-3i}\right| = \frac{|2+i|}{|1-3i|}$$

$$= \frac{\sqrt{5}}{\sqrt{10}} = \frac{\sqrt{2}}{2}$$

Interpretation of $|z_B - z_A|$

Consider two complex numbers $z_A = x_1 + iy_1$, and $z_B = x_2 + iy_2$. The points A and B represent z_A and z_B respectively.



Then, $z = (x_2 - x_1) + i(y_2 - y_1)$ and is represented by the point C. This makes $OABC$ a parallelogram.

From this, it follows that $|z_B - z_A| = \overline{OC} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

That is to say, $|z_B - z_A|$ is the length AB in the Argand diagram.

If the complex number z_A is represented by the point A, and the complex number z_B is represented by the point B,

then $|z_B - z_A| = \overline{AB}$,

Example 1.11

Let A and B be the points with affixes $z_A = -1 - i,\ z_B = 2 + 2i$.

Find $\overline{AB}$.

Solution

$$\overline{AB} = |z_B - z_A| = |3 + 3i| = \sqrt{9 + 9} = 3\sqrt{2}$$

Application activity 1.4

Find the modulus of each of the following complex numbers:

1) $2 + i$ 2) $-4 + 3i$ 3) $\frac{5}{3 - 4i}$

4) $\frac{2 + i}{1 - 3i}$ 5) $i + \frac{2}{1 + i}$ 6) $(3 + 4i)(2 - i)$

Loci related to the distances on Argand diagram

A locus is a path traced out by a point subjected to certain restrictions. Paths can be traced out by points representing variable complex numbers on an Argand diagram just as they can in other coordinate systems.

Activity 1.5

Sketch the set of points determined by the condition

$|z - 1 + 3i| = 2$.

Hint: Replace z by $x + yi$ and perform other operations.

Consider the simplest case first, when the point P represents the complex number z such that $|z| = R$. This means that the distance of OP from the origin O is constant and so P will trace out a circle.

$|z| = R$ represents a circle with centre at origin and radius R.

If instead $|z - z_1| = R$, where z_1 is a fixed complex number represented by point A on Argand diagram, then

$|z - z_1| = R$ represents a circle with centre z_1 and radius R.

Note that if $|z - z_1| \leq R$, then the point P representing z cannot only lie on the circumference of the circle, but also anywhere inside the circle. The locus of P is therefore the region on and within the circle with centre A and radius R.

Now, consider the locus of a point P represented by the complex number z subjected to the conditions $|z - z_1| = |z - z_2|$,

where z_1 and z_2 are fixed complex numbers represented by the points A and B on an Argand diagram. Then

$|z - z_1| = |z - z_2|$ represents a straight line which is the perpendicular bisector (mediator) of the line segment joining the points z_1 and z_2.

Note also that if $|z - z_1| \leq |z - z_2|$, the locus of z is not only the perpendicular bisector of AB but also the whole half line in which A lies, bound by this bisector.

Example 1.12

If $\left|\dfrac{z+2}{z}\right| = 2$ and point P represent z in the Argand plane, show that P lies on a circle and find the centre and radius of this circle.

Solution

Let $z = x + iy$ where $x, y \in \mathbb{R}$

Then $\left|\dfrac{z+2}{z}\right| = 2$ $\qquad \Rightarrow |z+2| = 2|z|$

$\Rightarrow |x + iy + 2| = 2|x + iy|$ $\qquad \Rightarrow |x + 2 + iy| = 2|x + iy|$

$$\Rightarrow \sqrt{(x+2)^2 + y^2} = 2\sqrt{x^2 + y^2}$$

$$\Rightarrow (x+2)^2 + y^2 = 4x^2 + 4y^2 \qquad [\textit{squaring both sides}]$$

$\Rightarrow x^2+4x+4+y^2=4x^2+4y^2 \qquad \Rightarrow -3x^2-3y^2+4x=-4$

$\Rightarrow 3x^2+3y^2-4x=4$ which is the equation of a circle with centre at $\left(\frac{2}{3},0\right)$ and with radius of length $\frac{4}{3}$.

Example 1.13

Determine, in complex plane, the locus M of affix z such that $|z-2i|=|z+2|$

Solution

Let $z=x+yi$, we have

$|x+yi-2i|=|x+yi+2|$

$\Rightarrow |x+i(y-2)|=|x+2+yi|$

$\Rightarrow \sqrt{x^2+(y-2)^2}=\sqrt{(x+2)^2+y^2}$

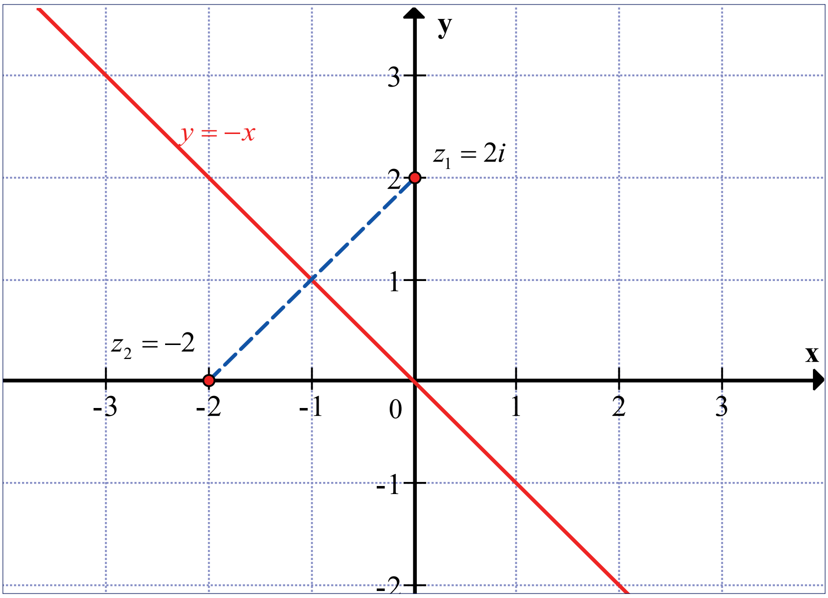
$\Rightarrow x^2+(y-2)^2=(x+2)^2+y^2 \qquad [\textit{squaring both sides}]$

$\Rightarrow x^2+y^2-4y+4=x^2+4x+4+y^2$

$\Rightarrow -4y=4x$

$\Rightarrow y=-x$

This is a straight line, mediator of the line segment joining the points $z_1=2i$ and $z_2=-2$. See the following figure.



Application activity 1.5

1. If $\left|\frac{2z+1}{z}\right| = 1$ and P represent z in the Argand plane, show that P lies on a circle and find the centre and radius of this circle.
2. Determine, in complex plane, the set of points M of affix z such that:

   a) $|z| = 2$ b) $|z| < 2$ c) $|z| > 2$

   d) $|z+1| = 1$ e) $|z+1| = |z-1|$ f) $|z-1+3i| = 2$

1.2.4. Operations on complex numbers

Equality of two complex numbers

Activity 1.6

1. Present in the same Argand diagram the following complex numbers

   $3+2i-1$ and $2+4i-2i$.

   What is your observations?

   From their real and imaginary parts, establish a condition for **Equality of two complex numbers**.

2. Find x if $x+2i = x+2xi-3i$.

3. Find x and y if $x+yi = 3y-(2x-4)i$.

If two complex numbers, say $a+bi$ and $c+di$ are equal, then their real parts are equal and their imaginary parts are equal. That is, $a+bi = c+di \Leftrightarrow a = c \text{ and } b = d$.

Example 1.14

Given $z = a+b-4i,\ w = 2+bi$. Find the values of a and b if $z = w$.

Solution

$z = w \Leftrightarrow a+b = 2, -4 = b$

$a-4 = 2 \Rightarrow a = 6$

Thus, $a = 6,\ b = -4$

Application activity 1.6

Find the values of x and y if:

1. $x+3i = 4-yi$
2. $x+(x+1)i = 5+yi$
3. $yi+x = 3i+y$
4. $yi-6 = x+9i$
5. $x+3+7i = (x+y)i+5$
6. $1+xi = 3i+y$
7. $x-4+6i = 2-yi$
8. $x-3-6i = (x-y)i+5$

Addition and subtraction

Activity 1.7

Let $z_1 = 2+3i$ and $z_2 = 5-4i$ be two complex numbers.

1. Evaluate z_1+z_2 and z_1-z_2.
2. State the real and imaginary parts of z_1+z_2 and z_1-z_2.

Consider the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ where $A(a,b), B(c,d)$ with $a,b,c,d \in \mathbb{R}$.

In fig. 1.3, $\overrightarrow{OX}$ is sum of the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

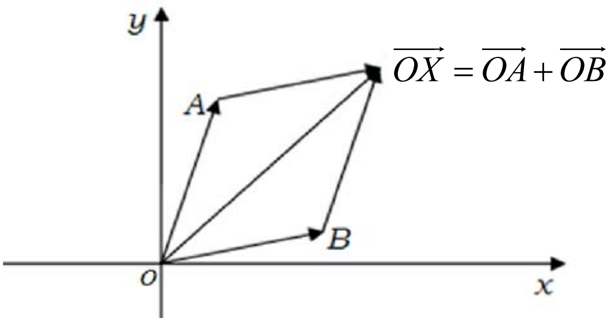


Figure. 1.3. *Addition of two complex numbers*

Addition or **subtraction** of two complex numbers can be done geometrically by constructing a parallelogram (see Fig 1.3).

From activity 1.7, two complex numbers are added (or subtracted) by adding (or subtracting) separately the two real and the two imaginary parts. That is to say,

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

$$(a+bi)-(c+di)=(a-c)+(b-d)i$$

Particular element:

$$\left.\begin{array}{l}(a,b)+(0,0)=(a,b)\\(0,0)+(a,b)=(a,b)\end{array}\right\} \Rightarrow (0,0) \text{ is an additive identity.}$$

Example 1.15

Evaluate $z_1 + z_2$ if $z_1 = 3+4i$ and $z_2 = 1+2i$

Solution

$$(3+4i)+(1+2i)=(3+1)+(4+2)i=4+6i$$

Example 1.16

Evaluate $z_1 - z_2$ if $z_1 = 1 - 2i$ and $z_2 = 9 + 3i$

Solution

$$(1-2i)-(9+3i)=(1-9)+(-2-3)i=-8-5i$$

Application activity 1.7

For each of the following pairs, evaluate $z_1 + z_2$ and $z_1 - z_2$.

1. $z_1 = 3i,\ z_2 = -12 - 3i$
2. $z_1 = 12i - 5,\ z_2 = 5 + 4i$
3. $z_1 = 3 + 4i,\ z_2 = 2 - i$
4. $z_1 = -23 - 14i,\ z_2 = 21 - 10i$
5. $z_1 = i,\ z_2 = -32i$
6. $z_1 = 10i + 3,\ z_2 = -5 - 2i$
7. $z_1 = 13 - 14i,\ z_2 = 22 + i$
8. $z_1 = 3 - i,\ z_2 = 1 + 10i$

Conjugate and opposite

Activity 1.8

Let $z_1 = 4 + 3i,\ z_2 = 4 - 3i$ and $z_3 = -4 - 3i$

1. Plot on Argand diagram complex numbers z_1, z_2 and z_3 and discuss their relationship.
2. Evaluate

   a) $\frac{1}{2}(z_1 + z_2)$ b) $\frac{1}{2i}(z_1 - z_2)$

3. Comment on your results in 2.

The **complex conjugate** of the complex number $z = x + yi$, denoted by $\bar{z}$ or z^*, is obtained by changing the sign of the imaginary part. Hence, the complex conjugate of $z = x + yi$ is $\bar{z} = x - yi$.

The complex number $-z = -x - yi$ is the **opposite** of $z = x + yi$, symmetric of z with respect to 0.

Geometrical presentation of conjugate and opposite (negative) of complex number

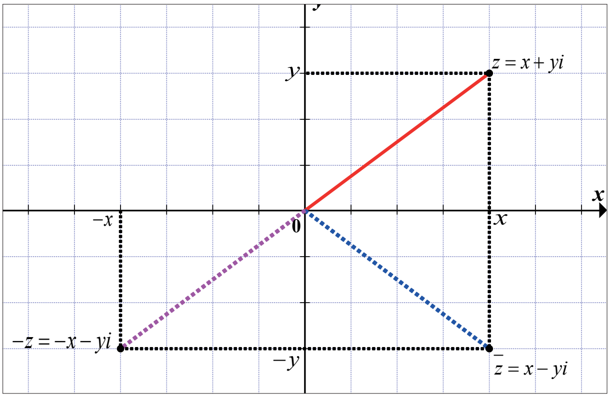


Figure 1.4. *Geometrical presentation of conjugate and opposite of a complex number*

Geometrically, figure 1.4 shows that $\overline{z}$ is the "reflection" of z about the real axis while $-z$ is symmetric to z with respect to 0. In particular, conjugating twice gives the original complex number: $\overline{\overline{z}} = z$.

The real and imaginary parts of a complex number can be extracted using the conjugate:

$$\operatorname{Re}(z) = \frac{1}{2}(z + \overline{z}) \qquad \operatorname{Im}(z) = \frac{1}{2i}(z - \overline{z})$$

Moreover, a complex number is real if and only if it equals its conjugate.

Example 1.17

Consider the complex number $z = 1 + 2i$. Show that $\operatorname{Re}(z) = \frac{1}{2}(z + \overline{z})$ and $\operatorname{Im}(z) = \frac{1}{2i}(z - \overline{z})$

Solution

$$z = 1 + 2i, \qquad \overline{z} = 1 - 2i$$

$$\frac{1}{2}(z + \overline{z}) = \frac{1}{2}(1 + 2i + 1 - 2i) = \frac{1}{2}(2) = 1 = \operatorname{Re}(z)$$

$$\frac{1}{2i}(z - \overline{z}) = \frac{1}{2i}(1 + 2i - 1 + 2i) = \frac{1}{2i}(4i) = 2 = \operatorname{Im}(z)$$

Notice

- Conjugation distributes over the standard arithmetic operations:

  (i) $\overline{z+w}=\overline{z}+\overline{w}$ (ii) $\overline{zw}=\overline{z}\,\overline{w}$ (iii) $\overline{\left(\dfrac{z}{w}\right)}=\dfrac{\overline{z}}{\overline{w}}$

  (iv) $\mathrm{Im}(z)=-Im(\overline{z})$ *and* $\mathrm{Re}(z)=\mathrm{Re}(\overline{z})$
- The complex number $-z=-x-yi$ is the opposite of $z=x+yi$, symmetric of z with respect to 0 and $z+|-z|=0$.
- $-z=-x-yi$ is the opposite of $z=x+yi$

  $z+(-z)=0$

Example 1.18

Find the conjugate of $z=u+w$, if $u=6+2i$ and $w=1-4i$.

Solution

Conjugate of $z=(6+2i)+(1-4i)$ is

$$\overline{z}=\overline{(6+2i)+(1-4i)}=\overline{7-2i}=7+2i$$

Or

$$\overline{z}=\left(\overline{6+2i}\right)+\left(\overline{1-4i}\right)=(6-2i)+(1+4i)=7+2i$$

Example 1.19

Find the conjugate of $z=u-w-t$, if $u=2-3i$, $w=-1-i$ and $t=4+3i$.

Solution

Conjugate of $z=(2-3i)-(-1-i)-(4+3i)$ is

$$\bar{z}=\overline{(2-3i)-(-1-i)-(4+3i)}=\overline{-1-5i}=-1+5i$$

Or

$$\bar{z}=\overline{(2-3i)}-\overline{(-1-i)}-\overline{(4+3i)}=(2+3i)-(-1+i)-(4-3i)=-1+5i$$

Application activity 1.8

Find the conjugate of:

1. $z=-76$
2. $z=-9i$
3. $z=12-4i$
4. $z=3+i$
5. $z=(-9+7i)+(1+3i)$
6. $z=(4+i)-(1+2i)$
7. $z=(4-6i)-(1-i)$
8. $z=(-2-2i)-(1-4i)+(3i-2)$

Multiplication

Activity 1.9

Let $z_1=2-3i$ and $z_2=3+2i$.

1. Evaluate $z_1 \times z_2$.
2. State the real and imaginary parts of $z_1 \times z_2$.

From activity 1.9, the **multiplication** of two complex numbers $z_1=a+bi$ and $z_2=c+di$ is defined by the following formula:

$$\begin{aligned} z_1 \times z_2 &= (a+bi)(c+di) \\ &= (ac-bd)+(bc+ad)i \end{aligned}$$

Alternatively, if $z_1(a,b), z_2(c,d)$ are complex numbers in geometric form, thus, $z_1 \cdot z_2 = (ac-bd, bc+ad)$.

In particular, the square of the imaginary unit is -1; since $i^2=i\times i=-1$ or in geometric form $(0,1)(0,1)=(-1,0)$.

The preceding definition of multiplication of general complex numbers follows naturally from this fundamental property of the imaginary unit.

Particular elements:

- $(a+bi)1=a+bi$ and $1(a+bi)=a+bi \Rightarrow 1$ is the multiplicative identity.
- $(a+bi)i=-b+ai$ and $i(a+bi)=-b+ai \Rightarrow i$ is the imaginary unit.

- $i \cdot i = -1 \Rightarrow i^2 = -1$ is the fundamental relation.

Application activity 1.9

For each of the following pairs, evaluate $z_1 \cdot z_2$

1. $z_1 = 3i,\ z_2 = -12 - 3i$
2. $z_1 = 12i - 5,\ z_2 = 5 + 4i$
3. $z_1 = 3 + 4i,\ z_2 = 2 - i$
4. $z_1 = i,\ z_2 = -10 + 3i$
5. $z_1 = 11i + 4,\ z_2 = -3 - 2i$
6. $z_1 = -3 - 2i,\ z_2 = 2 + i$

Inverse and division

Activity 1.10

Consider the complex numbers $z_1 = 2 + i$ and $z_2 = 3 - i$.

1. Evaluate $z_1 \cdot \overline{z_1}$. What conclusion do you draw from your result?
2. From result obtained in 1), find $\frac{1}{z_1}$.
3. Using the result obtained in 2), deduce the formula for $\frac{z_1}{z_2}$ supposing that $\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2}$

From activity 1.10, the **inverse** of $z = a + bi$ is given by

$\frac{1}{z} = z^{-1} = \frac{\overline{z}}{z \times \overline{z}}$ where $\overline{z} = a - bi$

Remark

The product $z\bar{z} = a^2 + b^2$ is called the **norm** of $z = a + bi$ and is denoted by $\|z\|^2$ or $|z|^2$.

Thus,

$\frac{1}{z} = \frac{\bar{z}}{\|z\|^2}$ where $\bar{z} = a - bi$

Hence,

$$z^{-1} = \frac{\bar{z}}{a^2 + b^2} = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$$

Also from activity 1.10, the **division** of two complex numbers is achieved by multiplying both numerator and denominator by the complex conjugate of the denominator.

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$\frac{z_1}{z_2} = \frac{z_1 \bar{z}_2}{\|z_2\|^2}$$

Or

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di} = \left(\frac{ac + bd}{c^2 + d^2}\right) + i\left(\frac{bc - ad}{c^2 + d^2}\right)$$

Example 1.20

Find $\frac{1}{z}$ if $z = 4 + 2i$

Solution

$$\frac{1}{z} = \frac{1}{4 + 2i} = \frac{4 - 2i}{4^2 + 2^2} = \frac{4}{20} - \frac{2}{20} i = \frac{1}{5} - \frac{1}{10} i$$

Example 1.21

Evaluate $\frac{-1 + i}{i + 2}$

Solution

$$\frac{-1+i}{i+2}=\frac{-1+i}{2+i}=\frac{(-1+i)(2-i)}{(2+i)(2-i)}=\frac{-2+1+2i+i}{1+4}=\frac{-1+3i}{5}.$$

Example 1.22

Find the real numbers x and y such that $(x+iy)(3-2i)=6-17i$.

Solution

$(x+iy)(3-2i)=6-17i$

$$\Rightarrow x+iy=\frac{6-17i}{3-2i}=\frac{(6-17i)(3+2i)}{(3-2i)(3+2i)}=\frac{52-39i}{13}=4-3i$$

Thus, $x=4$ and $y=-3$

Alternative method

$(x+iy)(3-2i)=6-17i$

$3x-2ix+3iy+2y=6-17i \Leftrightarrow 3x+2y+(-2x+3y)i=6-17i$

$$\Leftrightarrow \begin{cases} 3x+2y=6 \\ -2x+3y=-17 \end{cases}$$

Solving this system, we get;

$x=4$ and $y=-3$

Remarks

1. Three distinct points A, B and C with affixes z_1, z_2 and z_3 respectively are collinear if and only if $\frac{z_C-z_A}{z_B-z_A}\in IR$
2. The non-zero vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular if and only if $\frac{z_C-z_A}{z_B-z_A}$ is pure imaginary different from zero.

Example 1.23

Let A, B and C be the points with affixes $z_A = -1-i$, $z_B = 2+2i$ and $z_C = 3+3i$ respectively. Show that they are collinear points.

Solution

$$\frac{z_C - z_A}{z_B - z_A} = \frac{(3+3i)-(-1-i)}{(2+2i)-(-1-i)}$$

$$\frac{4+4i}{3+3i} = \frac{4(1+i)}{3(1+i)} = \frac{4}{3} \in IR$$

Thus, z_A, z_B and z_C are collinear.

Example 1.24

Let A, B and C be the points with affixes $z_A = 2+i$, $z_B = 3+2i$ and $z_C = 1+2i$ respectively.

Show that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular.

Solution

$z_C - z_A = \overrightarrow{AC} = -1+i \qquad z_B - z_A = \overrightarrow{AB} = 1+i$

$$\frac{Z_c - Z_A}{Z_B - Z_A} = \frac{-1+i}{1+i} = \frac{(-1+i)(1-i)}{(1+i)(1-i)} = \frac{2i}{2} = i$$

This is pure imaginary different from zero. Thus, the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are perpendicular.

Application activity 1.10

For each of the following pairs, evaluate $\frac{1}{z_1}, \frac{1}{z_2}, \frac{z_1}{z_2}$

1. $z_1 = 3i,\ z_2 = -12-3i$
2. $z_1 = 12i-5,\ z_2 = 5+4i$
3. $z_1 = 3+4i,\ z_2 = 2-i$
4. $z_1 = -23-14i,\ z_2 = 21-10i$
5. $z_1 = 1-3i,\ z_2 = -1+2i$
6. $z_1 = i-2,\ z_2 = -5+2i$
7. $z_1 = -1+i,\ z_2 = 1-i$
8. $z_1 = -2+12i,\ z_2 = 1+10i$

1.2.5. Square root of a complex number

Activity 1.11

Consider the complex number $z = 8-6i$. considering that $(x+yi)^2 = 8-6i$, Determine the values of x and y such that $x+yi$ is the square root of z.

In general, from activity 1.11, if a complex number $x+yi$ is a square root of the complex number $a+bi$, thus, $(x+iy)^2 = a+bi \Rightarrow |x+iy|^2 = |a+bi|$ and then

$$\begin{cases} x = \pm\sqrt{\frac{1}{2}\left(a+\sqrt{a^2+b^2}\right)} \\ y = \pm\sqrt{\frac{1}{2}\left(\sqrt{a^2+b^2}-a\right)} \end{cases}$$

Remark

If $b \neq 0$, from activity 1.11, the sign cannot be taken arbitrary because the product xy has sign of b. Then,

- If $b > 0$, we take the same sign.
- If $b < 0$, we interchange signs.

In each case, the complex number has two roots.

Example 1.25

Find square roots of the complex number $3+4i$.

Solution

Thus $\begin{cases} a = 3 \\ b = 4 > 0 \end{cases}$

$$\begin{cases} x = \pm\sqrt{\frac{1}{2}\left(\sqrt{25}+3\right)} = \pm\sqrt{4} = \pm 2 \\ y = \pm\sqrt{\frac{1}{2}\left(\sqrt{25}-3\right)} = \pm\sqrt{1} = \pm 1 \end{cases}$$

Since b is greater than zero, we take the same signs.

$\Rightarrow \sqrt{3+4i} = 2+i \quad or \quad \sqrt{3+4i} = -2-i$.

We can write $\sqrt{3+4i} = \pm(2+i)$

Alternative method

Let $z = x+iy$ be the square root of $3+4i$.

Thus, $(x+iy)^2 = 3+4i$ and $|z|^2 = |3+4i|$

Or $x^2 - y^2 + 2ixy = 3+4i$ and $x^2 + y^2 = \sqrt{9+16}$

Or $x^2 - y^2 + 2ixy = 3+4i$ and $x^2 + y^2 = 5$.

From the above two equations and equality of complex numbers, we have the simultaneous equations

$$\begin{cases} x^2 - y^2 = 3 \\ 2xy = 4 \\ x^2 + y^2 = 5 \end{cases}$$

1st and 3rd equations yield $2x^2 = 8$ and $2y^2 = 2$

$\Rightarrow x^2 = 4$ and $y^2 = 1$, which gives $x = \pm 2$ and $y = \pm 1$.

Since the product of x and y is positive, $z = 2+i$ or $z = -2-i$;

Hence, $\sqrt{3+4i} = \pm(2+i)$

Example 1.26

Find square roots of the complex number $-3-4i$.

Solution

Thus $\begin{cases} a = -3 \\ b = -4 < 0 \end{cases}$

$$\begin{cases} x = \pm\sqrt{\frac{1}{2}\left(\sqrt{25}-3\right)} = \pm\sqrt{1} = \pm 1 \\ y = \pm\sqrt{\frac{1}{2}\left(\sqrt{25}+3\right)} = \pm\sqrt{4} = \pm 2 \end{cases}$$

Since b is less than zero, we interchange the signs.

$\Rightarrow \sqrt{-3-4i} = 1-2i$ *or* $\sqrt{-3-4i} = -1+2i$.

We can write $\sqrt{-3-4i} = \pm(1-2i)$

Example 1.27

Find square roots of the complex number $-2i$.

Solution

$\begin{cases} a = 0 \\ b = -2 < 0 \end{cases}$ $\begin{cases} x = \pm\sqrt{\dfrac{2}{2}} = \pm 1 \\ y = \pm\sqrt{\dfrac{2}{2}} = \pm 1 \end{cases}$

As b is less than zero, we take the different signs.

$\Rightarrow \sqrt{-2i} = 1-i$ *or* $\sqrt{-2i} = -1+i$.

We can write $\sqrt{-2i} = \pm(1-i)$

Example 1.28

Find square roots of the complex number -2.

Solution

$\begin{cases} a = -2 < 0 \\ b = 0 \end{cases}$

$x = 0$ *and* $y = \pm\sqrt{2}$

$\Rightarrow \sqrt{-2} = \pm i\sqrt{2}$

Application activity 1.11

Find square roots of the following complex numbers:

1. $14i$ 2. $-24-10i$ 3. $-20+48i$ 4. $-91-60i$
5. $5+12i$ 6. $32-24i$ 7. $32+24i$ 8. $-119+120i$

1.2.6. Equations in the set of complex numbers

Linear equations

Activity 1.12

Find the value of z in;

1. $z+3i-4=0$
2. $4-i+iz=4z-3i$
3. $(1+i)(i+z)=4i$
4. $(1-i)z=2+i$

Recall that to solve the equation $3x+3=8$ is to find the value of x that satisfies the given equation. Here, we do the following:

$3x+3=8 \Rightarrow 3x=8-3$

$3x=5 \Rightarrow x=\frac{5}{3}$ and then the solution set is $S=\left\{\frac{5}{3}\right\}$.

In complex numbers also, we may need to find the complex number z that satisfies the given equation.

Example 1.29

Find the value of z if $4z+5i=12-i$.

Solution

$4z+5i=12-i$

$4z=12-i-5i$

$z=\frac{12-6i}{4} \Rightarrow z=3-\frac{3}{2}i$

Example 1.30

Find the value of z if $-1+i=(i+2)z$.

Solution

$-1+i=(i+2)z$

$$z=\frac{-1+i}{i+2}=\frac{(-1+i)(-i+2)}{(i+2)(-i+2)}=\frac{i-2+1+2i}{1+4}=\frac{-1+3i}{5}$$

Example 1.31

Solve $z=(1-i)\bar{z}+3+2i$, where $\bar{z}$ is the conjugate of z.

Solution

Let $z=x+yi \Rightarrow \bar{z}=x-yi$

$$x+yi=(1-i)(x-yi)+3+2i \Leftrightarrow x+yi=x-yi-ix-y+3+2i$$

$$\Leftrightarrow yi+yi+ix+y=3+2i \Leftrightarrow y+i(x+2y)=3+2i$$

$$\Rightarrow \begin{cases} y=3 \\ x+2y=2 \end{cases} \Rightarrow \begin{cases} x=-4 \\ y=3 \end{cases}$$

Thus, $z=-4+3i$ and $\bar{z}=-4-3i$

Application activity 1.12

Find the value of z in each of the following equations:

1. $2+3i-z=i$
2. $2-i-z=3iz+13i$
3. $(1-zi)(i+1)=14i$
4. $(1+3i)z=2i+4i$

Quadratic equations

Activity 1.13

Solve the following equations:

1. $x^2+2x+3=0$
2. $x^2+2x+1+i=0$

Recall that if $ax^2+bx+c=0$ then, $\Delta=b^2-4ac$

From activity 1.13, let a, b and c be real numbers ($a\neq 0$), then the equation $az^2+bz+c=0$ has either two real roots, one double real root or two conjugate complex roots.

Remarks

1. If $\Delta > 0$, there are two distinct real roots:

$$z_1 = \frac{-b+\sqrt{\Delta}}{2a} \text{ and } z_2 = \frac{-b-\sqrt{\Delta}}{2a}.$$

2. If $\Delta = 0$, there is a double real root:

$$z_1 = z_2 = -\frac{b}{2a}$$

3. If $\Delta < 0$, there is no real root. In this case, there are two conjugate complex roots:

$$z_1 = \frac{-b+i\sqrt{-\Delta}}{2a} \text{ and } z_2 = \frac{-b-i\sqrt{-\Delta}}{2a}.$$

Where

$$\Delta = b^2 - 4ac$$

$$z_1 + z_2 = -\frac{b}{a}, \quad z_1 \cdot z_2 = \frac{c}{a}$$

Example 1.32

Solve in $\mathbb{C}$: $z^2 + z + 1 = 0$

Solution

$$\Delta = 1 - 4 = -3 < 0$$

$$z_1 = \frac{-1+i\sqrt{3}}{2}, \quad z_2 = \frac{-1-i\sqrt{3}}{2}$$

Thus,

$$S = \left\{ \frac{-1-i\sqrt{3}}{2}, \frac{-1+i\sqrt{3}}{2} \right\}$$

Example 1.33

Solve in $\mathbb{C}$: $z^2 - 2(\cos\beta)z + 1 = 0, \quad \beta \in IR$

Solution

$$\Delta = 4\cos^2\beta - 4 = 4\left(\cos^2\beta - 1\right) = -4\sin^2\beta$$

$$z_1 = \frac{2\cos\beta - 2i\sin\beta}{2},\ z_2 = \frac{2\cos\beta + 2i\sin\beta}{2}$$

$$z_1 = \cos\beta - i\sin\beta,\ z_2 = \cos\beta + i\sin\beta$$

Thus,

$$S = \{\cos\beta - i\sin\beta, \cos\beta + i\sin\beta\}$$

Example 1.34

Solve in $\mathbb{C}: z^2 + \left(\sqrt{3}+i\right)z + 1 = 0$

Solution

$$\Delta = \left(\sqrt{3}+i\right)^2 - 4(1)(1) = 3 + 2\sqrt{3}\,i - 1 - 4 = -2 + 2\sqrt{3}\,i$$

$$\sqrt{\Delta} = \sqrt{-2+2\sqrt{3}\,i}$$

$$\sqrt{\Delta} = 1 + \sqrt{3}\,i \quad or \quad \sqrt{\Delta} = -1 - \sqrt{3}\,i$$

Now,

$$z_1 = \frac{-\left(\sqrt{3}+i\right) + \left(1+\sqrt{3}\,i\right)}{2} = \frac{1-\sqrt{3}+\left(-1+\sqrt{3}\right)i}{2}$$

$$z_2 = \frac{-\left(\sqrt{3}+i\right) - \left(1+\sqrt{3}\,i\right)}{2} = \frac{-1-\sqrt{3}-\left(1+\sqrt{3}\right)i}{2}$$

We could also use $\sqrt{\Delta} = -1 - \sqrt{3}\,i$:

$$z_1 = \frac{-\left(\sqrt{3}+i\right) + \left(-1-\sqrt{3}\,i\right)}{2} = \frac{-1-\sqrt{3}-\left(1+\sqrt{3}\right)i}{2}$$

$$z_2 = \frac{-\left(\sqrt{3}+i\right) - \left(-1-\sqrt{3}\,i\right)}{2} = \frac{1-\sqrt{3}+\left(-1+\sqrt{3}\right)i}{2}$$

Example 1.35

Solve in $\mathbb{C}: z^2 - iz = 4 + 2i$

Solution

$z^2 - iz = 4 + 2i$

$\Leftrightarrow z^2 - iz - 4 - 2i = 0$

$\Delta = (-i)^2 - 4(1)(-4-2i) = -1 + 16 + 8i = 15 + 8i$

$\sqrt{\Delta} = \pm(4+i)$

$z_1 = \dfrac{i+4+i}{2} = 2+i,\ z_1 = \dfrac{i-4-i}{2} = -2$

Hence, $S = \{2+i, -2\}$

Application activity 1.13

Solve, in complex numbers, the following equations:

1. $3z^2 + 10 = 4z$
2. $\dfrac{z^2}{2} = 5z - 17$
3. $z + \dfrac{5}{z} = 3$
4. $z^2 + z\sqrt{3} + 1 = 0$

1.2.7. Polynomials in set of complex numbers

Activity 1.14

1. Expand

   a) $(z-2-3i)(z+3+i)$ b) $(z-i)(z+3i)(z-4i)$

2. Show that $P(z) = z^3 + (-2-i)z^2 + (2+2i)z - 4$ is divisible by $z+i$ and hence, by using synthetic division, factorise completely $P(z)$.
3. Find the value of z such that $P(z) = 0$ if $P(z) = z^3 - 2z^2 + (7+2i)z - 6(2-i)$.

Given any complex number z and coefficients $a_0, \ldots, a_n$, then the equation $a_n z^n + \ldots. + a_1 z + a_0 = 0$ has at least one complex solution z, provided that at least one of the higher coefficients, $a_0, \ldots, a_n$, is non-zero.

The process of finding the roots of a polynomial in set of complex numbers is similar for the case of real number remembering that the square root of a negative real number exist in set of complex numbers considering $\sqrt{-1} = i$.

We need the important result known as the Fundamental Theorem of Algebra.

Fundamental Theorem of Algebra

Every polynomial of positive degree with coefficients in the system of complex numbers has a zero in the system of complex numbers. Moreover, every such polynomial can be factored linearly in the system of complex numbers.

Example 1.36

Factorise the polynomial $P(z) = 3z^4 - 5z^3 + 5z^2 - 5z + 2$

Solution

First, we need to find the values of z satisfying $3z^4 - 5z^3 + 5z^2 - 5z + 2 = 0$

Fundamental theorem states that a 4^{th} degree function has 4 roots.

$z = 1$ is zero (one of the roots) of $P(z)$ since $P(1) = 3 - 5 + 5 - 5 + 2 = 0$

$$\Rightarrow 3z^4 - 5z^3 + 5z^2 - 5z + 2 = (z-1)(3z^3 - 2z^2 + 3z - 2)$$

Factorising R.H.S further, we get

$$3z^4 - 5z^3 + 5z^2 - 5z + 2 = (z-1)\left[3z(z^2+1) - 2(z^2+1)\right] = (z-1)(z^2+1)(3z-2)$$

The factor $z^2 + 1$ is factorised as follows:

$$z^2 + 1 = 0 \Rightarrow z^2 = -1 \Rightarrow z = i \quad \text{or} \quad z = -i$$
$$\Rightarrow z^2 + 1 = (z-i)(z+i)$$

Thus, $P(z) = (z-1)(z-i)(z+i)(3z-2)$

Example 1.37

Solve the equation $z^4 - 1 = 0$.

Solution

To solve the equation $z^4 - 1 = 0$ is the same as to find 4^{th} (fourth) roots of unity, as we can write $z^4 = 1$.

Or we can work as follows:

Factorise $z^4 - 1$ in order to find the roots.

$z = 1$ is one root, then $z^4 - 1 = (z-1)(z^3 + z^2 + z + 1)$.

Factorising R.H.S further gives

$$z^4 - 1 = (z-1)\left[z(z^2+1) + (z^2+1)\right] = (z-1)(z^2+1)(z+1)$$

Thus,

$z^4 - 1 = 0$

$\Leftrightarrow z - 1 = 0 \Rightarrow z = 1$ or

$z^2 + 1 = 0 \Rightarrow z = i$ or $z = -i$ or $z + 1 = 0 \Rightarrow z = -1$

Then, $S = \{-1, 1, -i, i\}$.

Example 1.38

Factorise completely $P(z) = z^4 - 4(1+i)z^3 + 12iz^2 - 8i(1+i)z - 5$ and hence solve the equation $P(z) = 0$.

Solution

Factors of -5 are $\pm 1, \pm 5$

$$P(1) = 1^4 - 4(1+i)(1^3) + 12i(1^2) - 8i(1+i)(1) - 5$$
$$= 1 - 4 - 4i + 12i - 8i + 8 - 5 = 0$$

Then, 1 is one of the four roots.

Using Synthetic division, we have

	1	$-4-4i$	$12i$	$-8i+8$	-5
1		1	$-3-4i$	$-3+8i$	5
	1	$-3-4i$	$-3+8i$	5	0

Now $P(z)$ becomes;

$$P(z) = z^4 - 4(1+i)z^3 + 12iz^2 - 8i(1+i)z - 5$$
$$= (z-1)\left(z^3 + (-3-4i)z^2 + (3-8i)z + 5\right)$$

i is also a root

Again using synthetic approach we have;

	1	$-3-4i$	$-3+8i$	5
i		i	$3-3i$	-5
	1	$-3-3i$	$5i$	0

$P(z)$ becomes;

$$P(z)=z^4-4(1+i)z^3+12iz^2-8i(1+i)z-5$$
$$=(z-1)(z-i)\left(z^2+(-3-3i)z+5i\right)$$

Now, $z^2+(-3-3i)z+5i$ is factorised as follows:

$$\Delta=(-3-3i)^2-20i=9+18i-9-20i=-2i \Rightarrow \sqrt{\Delta}=\sqrt{-2i}=1-i \quad or -1+i$$

$$z=\frac{3+3i+1-i}{2}=\frac{4+2i}{2}=2+i \text{ or } z=\frac{3+3i-1+i}{2}=\frac{2+4i}{2}=1+2i$$

Then, $z^2+(-3-3i)z+5i=(z-1-2i)(z-2-i)$ and

$$P(z)=z^4-4(1+i)z^3+12iz^2-8i(1+i)z-5$$
$$=(z-1)(z-i)(z-1-2i)(z-2-i)$$

Now, $P(z)=0 \Leftrightarrow (z-1)(z-i)(z-1-2i)(z-2-i)=0$

$z-1=0 \Rightarrow z=1$

$z-i=0 \Rightarrow z=i$

$z-1-2i=0 \Rightarrow z=1+2i$

$z-2-i=0 \Rightarrow z=2+i$

Hence, $S=\{1,i,1+2i,2+i\}$

Application activity 1.14

1. Factorise the polynomials given below and hence find the zeros of each polynomial.

   a) $P(z)=z^3+6z+20$

   b) $Q(z)=z^3-6z^2+13z-10$

c) $R(z)=z^4-z^2+44z+26$

d) $M(z)=z^3+(-1-i)z^2+(10-2i)z-24+8i$

2. Find the values of the real numbers a and b if the complex number $5-2i$ is a zero of the polynomial $f(z)=2z^4-22z^3+az^2-68z+b$.

3. The polynomial $p(z)$ has degree 3. Given that $p(1+2i)=0, p(2)=0$ and $p(0)=20$, write $p(z)$ in the form az^3+bz^2+cz+d.

4. Solve the equation $(1+i)z^2+(3-2i)z-21+7i=0$

1.3. Polar form of a complex number

1.3.1. Argument of a complex number

Activity 1.15

Let; $z_1=1+i,\ z_2=1-i,\ z_3=-1+i,\ z_4=-1-i,\ z_5=i$ and $z_6=-i$.

Present in Argand diagram, affix of each given complex number and determine the value of angle θ for which affix of $z_k, k=1,2,\ldots,6$ makes with positive direction of real axis.

Hint: Use the definition; $\tan\theta_k=\dfrac{\operatorname{Im}(z_k)}{R_e(z_k)}$.

An alternative way of defining points in the complex plane, other than using the x and y coordinates, is to use the distance of a point P from the origin together with the angle between the line through P and O and the positive part of the real axis. This idea leads to the polar form of complex numbers.

The **argument** or **phase** θ (or amplitude) of z is the angle that the radius r makes with the positive real axis, as illustrated in figure 1.5, and is written as $\arg(z)$.

As with the modulus, the argument can be found from the rectangular form $x+yi$.

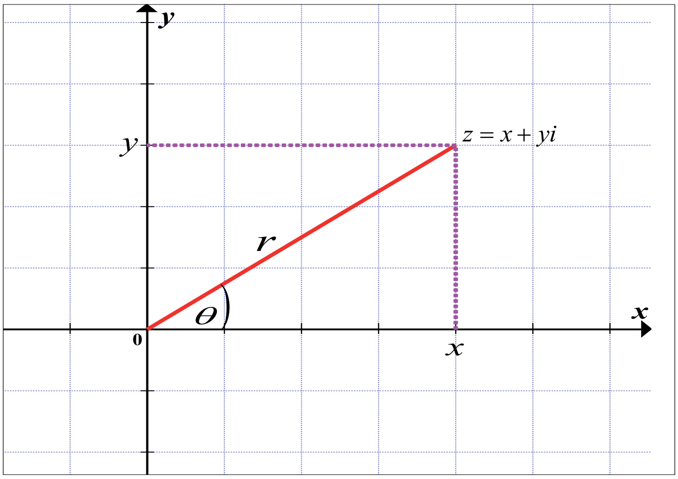


Figure 1.5. *Modulus and argument of a complex number*

Generally,

$$\arg(z)=\begin{cases}\arctan\dfrac{y}{x}, & if\ \ x>0\\ \pi+\arctan\dfrac{y}{x}, & if\ \ x<0, y\geq 0\\ -\pi+\arctan\dfrac{y}{x}, & if\ \ x<0,\ y<0\\ \dfrac{\pi}{2}, & if\ \ x=0,\ y>0\\ -\dfrac{\pi}{2}, & if\ \ x=0,\ y<0\\ Undefined & if\ \ x=0\, and\ y=0\end{cases}$$

The value of $\arg(z)$ must always be expressed in radians. It can change by any multiple of 2π and still give the same angle. Hence, the arg function is sometimes considered as multivalued. Normally, as given above, the **principal argument** in the interval $(-\pi,\pi]$ is chosen. The polar angle for the complex number 0 is undefined.

Example 1.39

Find the principal argument of the complex number $z=1+i$

Solution

$x = 1 > 0$

$$\arg(z) = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

Example 1.40

Find the principal argument of the complex number $z = 1 - i$

Solution

$x = 1 > 0$

$$\arg(z) = \arctan\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$$

Example 1.41

Find the principal argument of the complex number $z = -1 + i\sqrt{3}$

Solution

$x = -1 < 0, y = \sqrt{3} > 0$

$$\arg(z) = \pi + \arctan\left(\frac{\sqrt{3}}{-1}\right) = \pi + \left(-\frac{\pi}{3}\right) = \frac{3\pi - \pi}{3} = \frac{2\pi}{3}$$

Example 1.42

Find the principal argument of the complex number $z = -9\sqrt{3} - 9i$

Solution

$x = -9\sqrt{3} < 0, y = -9 < 0$

$$\arg(z) = -\pi + \arctan\left(\frac{-9}{-9\sqrt{3}}\right) = -\pi + \frac{\pi}{6} = \frac{-6\pi + \pi}{6} = \frac{-5\pi}{6}$$

Example 1.43

Find the principal argument of the complex number $z = 2i$

Solution

$x = 0, y > 0$

$\arg(z) = \frac{\pi}{2}$

Notice

The following relations are true:

- $\cos(\arg z) = \frac{x}{\sqrt{x^2 + y^2}}$
- $\sin(\arg z) = \frac{y}{\sqrt{x^2 + y^2}}$
- $\tan(\arg z) = \frac{y}{x}$

Let M be a point different from 0 with affix z. We denote θ, the argument of z.

- The symmetric of M with respect to the real axis has affix $\bar{z}$ with argument $-\theta$.
- The argument of the affix of symmetric of M with respect to 0 is $\theta + \pi$.

Application activity 1.15

Find the principal argument of the following complex numbers:

1. $14 - 14i$
2. $9\sqrt{3} - 9i$
3. $2 + 2i$
4. $3\sqrt{3} - 9i$
5. $6 - 2i\sqrt{3}$

Loci related to the angles on Argand diagram

Activity 1.16

Let $z = x + yi$. In Argand diagram, sketch the loci satisfying the conditions,

1. $\arg(z) = \frac{\pi}{4}$
2. $\arg(z - 4) = \frac{\pi}{3}$

The simplest case is the locus of P subject to the condition that $\arg(z)=\theta$, where θ is a fixed angle.

This condition implies that the angle between OP and Ox is fixed so that the locus of P is a straight line.

$\arg(z)=\theta$ represents the half line through O inclined at an angle θ to the positive direction of $x-axis$.

Note that the locus of P is only a half line; the other half line, shown dotted in figure 1.6, would have the equation $\arg(z)=\theta+\pi$, possibly add or subtract 2π if $\theta+\pi$ falls outside the specified range for $\arg(z)$.

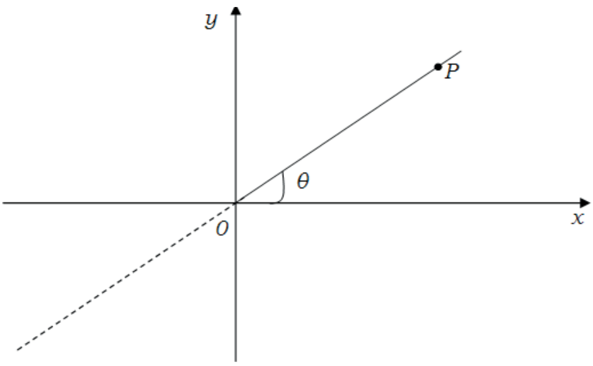


Figure 1.6. *Locus as a half line through 0*

Similarly, the locus of a point P satisfying $\arg(z-z_1)=\theta$, where z_1 is the affix of a fixed point A, is a line through A.

$\arg(z-z_1)=\theta$ represents the half line through the point z_1 inclined at an angle θ to the positive direction of $x-axis$.

Note again that this locus is only a half line; the other half line would have the equation $\arg(z-z_1)=\theta+\pi$, possibly adding or subtracting 2π if $\theta+\pi$ falls outside the specified range for $\arg(z)$ (as illustrated in the figure 1.7).

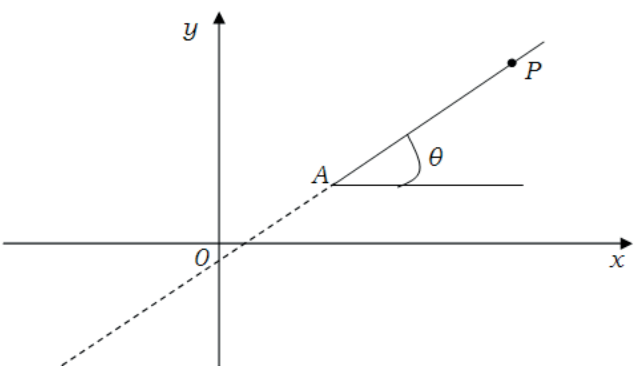


Figure 1.7. *Locus as a half line through point different from zero*

Finally, consider the locus of any point satisfying $\theta \le \arg(z - z_1) \le \beta$. This indicates that the angle between AP and the positive x-axis lies between θ and β, so that P can lie on or within the two half lines as shown in the figure 1.8.

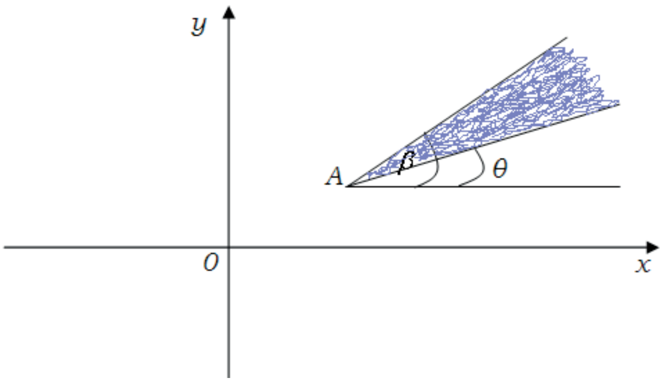


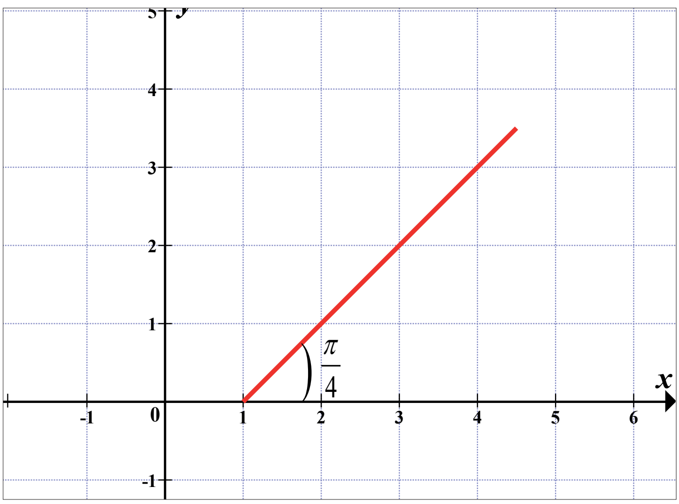
Figure 1.8. *Locus lies between two half lines*

Example 1.44

Sketch on Argand diagram the region where $\arg(z-1) = \dfrac{\pi}{4}$

Solution

At point (1,0), we trace a half-line inclined at an angle $\dfrac{\pi}{4}$ to the positive direction of x-axis. The needed region is given by all points lying on that half-line.



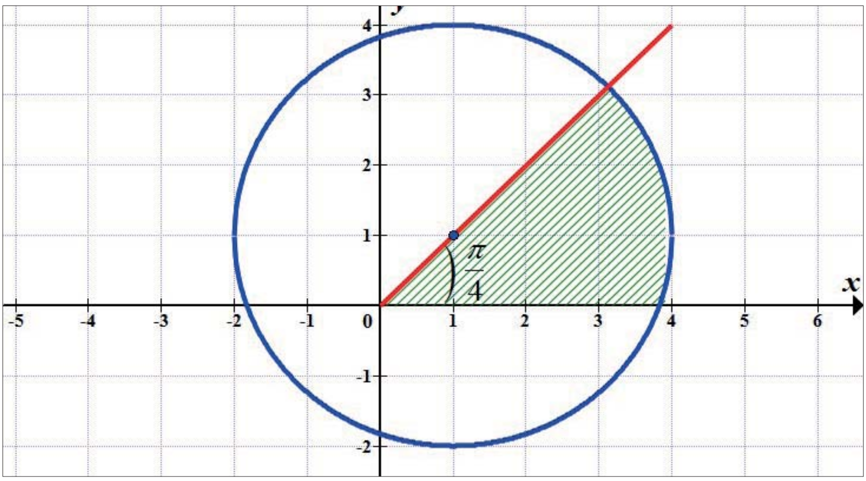
Example 1.45

Sketch on Argand diagram the region satisfying both $|z-1-i| \le 3$ and $0 \le \arg z \le \frac{\pi}{4}$

Solution

First $|z-1-i| \le 3$ is the region on and within the circle of centre $1+i$ and radius 3.

$0 \le \arg z \le \frac{\pi}{4}$ represents all points whose arguments are between or equal to 0 and $\frac{\pi}{4}$.



The required region is the shaded part.

Application activity 1.16

1. Sketch the loci satisfying these equations:

   a) $\arg(z-2-i) = -\frac{\pi}{4}$ b) $\arg(z-4) = \frac{\pi}{2}$

2. On the same Argand plane, sketch the loci of points satisfying:

   $$|z+3+i| = 5;\ \arg(z+3) = -\frac{3\pi}{4}$$

   a) From your sketch, explain why there is only one complex number satisfying both conditions.
   b) Verify that the complex number is $-7-4i$.

1.2.2. Polar form of a complex number

Activity 1.17

Given complex number $z = 5 + i$

1. Plot z in the Argand diagram.
2. Find modulus, r of z.
3. Find the angle, θ the line segment oz makes with the positive real axis.
4. From trigonometry and using the graph obtained in 1), express z in terms of r and θ .

From activity 1.17, r and θ can be calculated.

Generally, if r and θ are the modulus and principal argument

of complex number z respectively, then $z = r(\cos\theta + i\sin\theta)$.

This form is called **polar form** or **modulus-argument form** or trigonometric form of a complex number z .

$z = r(\cos\theta + i\sin\theta)$ is sometimes abbreviated as $z = r\,\text{cis}\theta$.

Where $cis\theta$ indicates $\cos\theta + i\sin\theta$ (or c for cosine, i for imaginary unit and s to denote sine).

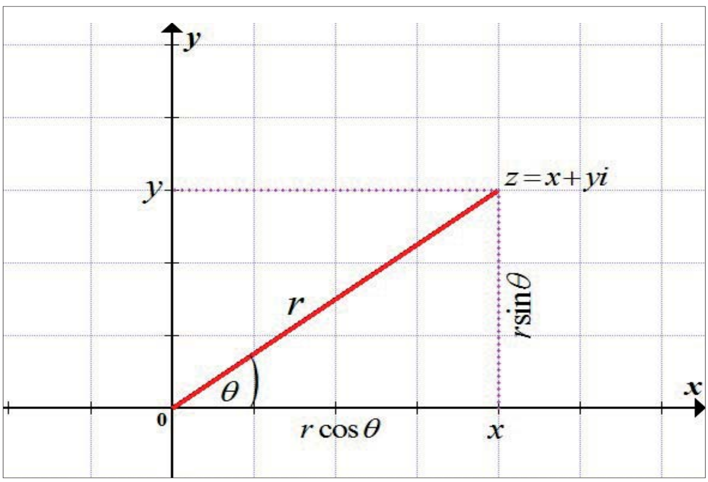


Figure 1.9. *Change of a complex number from algebraic to polar form*

From figure 1.9 and definition of trigonometric ratios, we have

$$\begin{cases} \sin\theta = \dfrac{y}{r} \\ \cos\theta = \dfrac{x}{r} \end{cases} \Leftrightarrow \begin{cases} y = r\sin\theta \\ x = r\cos\theta \end{cases}$$

And then, $z = x + iy \Rightarrow z = r\cos\theta + ir\sin\theta$ or $z = r(\cos\theta + i\sin\theta)$.

For brevity, $r(\cos\theta + i\sin\theta)$ can be written as (r,θ) or $r\angle\theta$.

Properties

1. Two complex numbers are equal if their modulii are equal and the difference between their argument is a multiple of 2π.

   If $z = r\,\mathrm{cis}\,\theta$ and $z' = r'\mathrm{cis}\theta'$

   Then, $z = z' \Leftrightarrow \begin{cases} r = r' \\ \theta = \theta' + 2k\pi \quad k \in \mathbb{Z} \end{cases}$

2. If $z = r(\cos\theta + i\sin\theta)$, its conjugate is $\overline{z} = r(\cos\theta - i\sin\theta)$ and its opposite is $-z = r(\cos(\theta+\pi) + i\sin(\theta+\pi))$.

Example 1.46

Express the complex number $z = 1 + i$ in polar form.

Solution

$|z| = \sqrt{2}$

$\arg(z) = \arctan 1 = \dfrac{\pi}{4}$

Polar form of z is

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Or $z = \sqrt{2}\mathrm{cis}\dfrac{\pi}{4}$ or $\sqrt{2}\angle\dfrac{\pi}{4}$

Example 1.47

Express the complex number $w = -\sqrt{2} + i\sqrt{2}$ in polar form.

Solution

$$|z| = 2,\ \arg(z) = \pi + \arctan(-1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Polar form: Or

$$z = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) \qquad z = 2\,\mathrm{cis}\,\frac{3\pi}{4}$$

Notice

Having a polar form of a complex number, you can get its corresponding algebraic form.

Example 1.48

Convert $\mathrm{cis}\left(-\frac{\pi}{3}\right)$ in algebraic form.

Solution

$$\mathrm{cis}\left(-\frac{\pi}{3}\right) = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Example 1.49

Convert $2\,\mathrm{cis}\left(-\frac{\pi}{2}\right)$ in algebraic form.

Solution

$$\begin{aligned} 2\,\mathrm{cis}\left(-\frac{\pi}{2}\right) &= 2\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right) \\ &= 2(0 - i) = -2i \end{aligned}$$

Application activity 1.17

1. Express the following complex numbers in polar form.

   a) 4 b) $2i$ c) -2 d) $-5i$

   e) $\sqrt{3}+i$ f) $\sqrt{3}-3i$ g) $3-i\sqrt{3}$

2. Convert the following complex numbers in cartesian form.

   a) $2\,\text{cis}\left(\frac{\pi}{3}\right)$ b) $4\,\text{cis}\frac{3\pi}{4}$ c) $\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

   d) $3\,\text{cis}\left(\frac{\pi}{2}\right)$ e) $4\,\text{cis}\,\pi$ f) $\text{cis}\left(-\frac{5\pi}{6}\right)$ g) $2\,\text{cis}\left(-\frac{\pi}{6}\right)$

1.2.3. Multiplication and division in polar form

Activity 1.18

Given two complex numbers $z_1 = 1-i$ and $z_2 = \sqrt{3}-i$,

1. Express $z_1 z_2$ in algebraic form and hence in polar form.
2. Express z_1 and z_2 in polar form and hence evaluate $z_1 z_2$.
3. What can you say about result in 1) and 2)?
4. Express $\frac{z_1}{z_2}$ in algebraic form and hence in polar form.
5. Using the polar forms of z_1 and z_2 in 2), evaluate $\frac{z_1}{z_2}$.
6. What can you say about the result in 4) and 5)?

Given two complex numbers $z_1 = r_1\left(cos\ \theta_1 + isin\ \theta_1\right)$ and $z_2 = r_2\left(cos\ \theta_2 + isin\ \theta_2\right)$, from activity 1.18, the formula for multiplication is

$$z_1 z_2 = r_1 r_2\left(cos\ \left(\theta_1+\theta_2\right) + isin\ \left(\theta_1+\theta_2\right)\right)$$

Provided that 2π must be added to or subtracted from $\theta_1+\theta_2$ if $\theta_1+\theta_2$ is outside the permitted range of the principal argument.

Similarly, division is given by

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}\left(\cos\left(\theta_1-\theta_2\right)+i\sin\left(\theta_1-\theta_2\right)\right)$$

With the provision that 2π may have to be added to, or substracted from $\theta_1-\theta_2$ if $\theta_1-\theta_2$ is outside the permitted range of the principal argument.

Example 1.50

Determine the product of the complex numbers $z=1+i$ and $w=\sqrt{3}+i$ in both cartesian and polar forms.

Solution

Cartesian form:

$$\begin{aligned}zw&=(1+i)\left(\sqrt{3}+i\right)\\&=\sqrt{3}+i+i\sqrt{3}-1\\&=\sqrt{3}-1+i\left(1+\sqrt{3}\right)\end{aligned}$$

Polar form:

$$z=\sqrt{2}\operatorname{cis}\frac{\pi}{4},\ w=2\operatorname{cis}\frac{\pi}{6}$$

$$zw=2\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$$

$$=2\sqrt{2}\operatorname{cis}\frac{3\pi+2\pi}{12}=2\sqrt{2}\operatorname{cis}\frac{5\pi}{12}$$

Alternatively, the polar form of zw can be determined as follows:

$$zw=\sqrt{3}-1+i\left(1+\sqrt{3}\right)$$

$$|zw|=\sqrt{\left(\sqrt{3}-1\right)^2+\left(1+\sqrt{3}\right)^2}=\sqrt{3-2\sqrt{3}+1+1+2\sqrt{3}+3}=\sqrt{8}=2\sqrt{2}$$

$$\arg(zw)=\arctan\frac{1+\sqrt{3}}{\sqrt{3}-1}=\frac{5\pi}{12}$$

Hence,

$$zw=2\sqrt{2}\operatorname{cis}\frac{5\pi}{12}$$

Example 1.51

Consider the complex number $z=\dfrac{\left(1-i\sqrt{3}\right)^4}{(1+i)^3}$.

a) Express z in algebraic form and polar form.

b) Deduce the exact value of $\cos\dfrac{\pi}{12}$ and $\sin\dfrac{\pi}{12}$.

Solution

a) $z = \dfrac{\left(1-i\sqrt{3}\right)^4}{\left(1+i\right)^3}$

$$= \frac{1-4i\sqrt{3}-18+12i\sqrt{3}+9}{1+3i-3-i} = \frac{-8+i8\sqrt{3}}{-2+2i} = \frac{-4+i4\sqrt{3}}{-1+i}$$

$$= \frac{\left(-4+i4\sqrt{3}\right)\left(-1-i\right)}{2} = \frac{4+4i-i4\sqrt{3}+4\sqrt{3}}{2}$$

$$= 2+2\sqrt{3}+i\left(2-2\sqrt{3}\right)$$

Thus, the algebraic form of z is $z = 2+2\sqrt{3}+i\left(2-2\sqrt{3}\right)$

Now,

$$|z| = \sqrt{\left(2+2\sqrt{3}\right)^2+\left(2-2\sqrt{3}\right)^2} = 4\sqrt{2}$$

$$\theta = \arg(z) = \arctan\frac{2+2\sqrt{3}}{2-2\sqrt{3}} = -\frac{\pi}{12}$$

Alternatively, z in polar form can be obtained as follows:

Let $z_1 = \left(1-i\sqrt{3}\right)^4$ and $z_2 = \left(1+i\right)^3$

$$z_1 = \left(1-i\sqrt{3}\right)^4$$

$$|z_1| = \left(\sqrt{1+3}\right)^4 = 16$$

$$\arg(z_1) = 4\times\arctan\left(-\sqrt{3}\right) = -\frac{4\pi}{3}$$

$$z_2 = \left(1+i\right)^3$$

$$|z_2| = \left(\sqrt{1+1}\right)^3 = 2\sqrt{2}$$

$$\arg(z_2) = 3\times\arctan(1) = \frac{3\pi}{4}$$

$$\Rightarrow |z| = \frac{|z_1|}{|z_2|} = \frac{16}{2\sqrt{2}} = \frac{16\sqrt{2}}{4} = 4\sqrt{2}$$ and

$$\theta = \arg(z)$$

$$= \arg(z_1) - \arg(z_2) = -\frac{4\pi}{3} - \frac{3\pi}{4} = \frac{-16-9\pi}{12} = -\frac{25\pi}{12}$$

Since we need the principal argument in the interval $(-\pi, \pi]$, we take $-\frac{25\pi}{12} + 2\pi$ because the value is large, negative and is not in the desired interval.

This gives $\frac{-25\pi + 24\pi}{12} = -\frac{\pi}{12}$, thus, $\arg(z) = -\frac{\pi}{12}$.

Then, the polar form of z is

$$z = 4\sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right)$$

$$= 4\sqrt{2}\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)$$

$$\left\| \begin{array}{l} \cos(-\alpha) = \cos\alpha \\ \sin(-\alpha) = -\sin\alpha \end{array} \right.$$

To deduce the exact value of $\cos\frac{\pi}{12}$ and $\sin\frac{\pi}{12}$, we equate the polar form of z and algebraic form of z:

$$4\sqrt{2}\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right) = 2 + 2\sqrt{3} + i\left(2 - 2\sqrt{3}\right)$$

$$\begin{cases} 4\sqrt{2}\cos\frac{\pi}{12} = 2 + 2\sqrt{3} \\ -4\sqrt{2}\sin\frac{\pi}{12} = 2 - 2\sqrt{3} \end{cases}$$

$$\Rightarrow \begin{cases} \cos\frac{\pi}{12} = \frac{2+2\sqrt{3}}{4\sqrt{2}} = \frac{2\sqrt{2}+2\sqrt{2}\sqrt{3}}{4\times 2} = \frac{\sqrt{2}+\sqrt{6}}{4} \\ \sin\frac{\pi}{12} = \frac{2-2\sqrt{3}}{-4\sqrt{2}} = \frac{2\sqrt{2}-2\sqrt{2}\sqrt{3}}{-4\times 2} = \frac{-\sqrt{2}+\sqrt{6}}{4} \end{cases}$$

Thus,

$$\begin{cases}\cos\dfrac{\pi}{12}=\dfrac{\sqrt{6}+\sqrt{2}}{4}\\ \sin\dfrac{\pi}{12}=\dfrac{\sqrt{6}-\sqrt{2}}{4}\end{cases}$$

Notice

Addition and subtraction in polar form of complex number is not possible directly as it is the case in multiplication and division. For addition and subtraction of complex numbers to be possible, each complex number has to be converted in to Cartesian form first.

Application activity 1.18

1. In each of the following , express zw and $\dfrac{z}{w}$ in polar form.

a) $z=1+i,\ w=-\sqrt{3}+i$ b) $z=2-2i\sqrt{3},\ w=cis\dfrac{2\pi}{3}$

c) $z=1+i,\ w=\sqrt{3}+i$ d) $z=2-2i,\ w=-\sqrt{3}+3i$

e) $z=-\dfrac{1}{2}-\dfrac{1}{2}i\sqrt{3},\ w=-1+i$

f) $z=i-1,\ w=i+1$

2. Express $\dfrac{(-1+i)\left(\sqrt{3}-3i\right)}{-3+i\sqrt{3}}$ in polar form.

1.2.4. Powers in polar form

Activity 1.19

Given a complex number $z=\sqrt{3}+i$;

1. Express z in polar form.
2. Given that $z^2=z\cdot z$, find the expression for z^2 in algebraic and in polar form.
3. Find the expression for $z^3=z^2\cdot z$ in algebraic form and hence in polar form.

4. Using results from 1 to 3, deduce the expression for z^n in polar form.

From activity 1.19, Power of a complex number $z = r(\cos\theta + i\sin\theta)$ is given by:

$$z^n = \left(r\left(\cos\theta + i\sin\theta\right)\right)^n = r^n\left(\cos n\theta + i\sin n\theta\right);\ n \in \mathbb{Z}_0$$

Theorem 1.1: De Moivre's theorem

From the power of a complex number, if $r = 1$, we have De Moivre's theorem:

$\left(\cos\theta + i\sin\theta\right)^n = \left(\cos n\theta + i\sin n\theta\right)$, for positive and negative integers and fractional values of n.

Application activity 1.19

1. Express the following in Cartesian form:

a) $\left(\frac{1}{2}+\frac{1}{2}i\sqrt{3}\right)^{10}$ b) $\left(\sqrt{3}+i\right)^{6}$ c) $(1-i)^{10}$

d) $\left(\frac{1-i\sqrt{3}}{2}\right)^{8}$ e) $\left(\frac{1+i}{\sqrt{2}}\right)^{12}$ f) $\left(1+i\sqrt{3}\right)^{9}$

g) $(2+2i)^{5}$

2. Find the positive integers m for which $\left(\sqrt{3}+i\right)^m - \left(\sqrt{3}-i\right)^m = 0$

1.2.5. n^{th} roots of a complex number

Activity 1.20

Given z=4

1. Express z in polar form.
2. Let $z_k = r'\text{cis}\,\theta'$ for $k = 0,1,2,3$ be the four 4^{th} roots of z. Using result in 1) and the expression $(z_k)^4 = z$, find all four 4^{th} roots of z in polar form.

From activity 1.20,

If $(z_k)^n = z$ for $z = rcis\theta$, then

$$z_k = \sqrt[n]{r}\, cis\left(\frac{\theta + 2k\pi}{n}\right) \qquad k = 0,1,2,3,\ldots\ldots,n-1$$

Here, $\sqrt[n]{r}$ is the usual (positive) n^{th} root of the positive real number r.

Example 1.52

Determine the 4^{th} roots of -4

Solution

$|-4| = 4$ and $\arg(-4) = \pi + \arctan 0 = \pi$

$$\Rightarrow z_k = \sqrt[4]{4}\left(cis\frac{\pi + 2k\pi}{4}\right);\ k = 0,1,2,3$$

$$z_0 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1 + i$$

$$z_1 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -1 + i$$

$$z_2 = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \sqrt{2}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -1 - i$$

$$z_3 = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = 1 - i$$

Then, the 4^{th} roots of -4 are $1+i, -1+i, -1-i,$ and $1-i$.

Special case: n^{th} roots of unity

Here, $z = 1$ and $|z| = 1$, $\arg(z) = 0$

Then,

$$z_k = \sqrt[n]{1}\ \text{cis}\frac{0 + 2k\pi}{n} = \text{cis}\frac{2k\pi}{n}$$

And then, the n^{th} roots of unity are given by

$$z_k = \text{cis}\frac{2k\pi}{n};\ k = 0,1,2,3,\ldots\ldots,n-1$$

This shows that the first root among the n^{th} roots of unity is always 1.

Notice

1. The n^{th} roots of unity can be used to find the n^{th} roots of any complex number if one of these roots is known.

   If one of the n^{th} roots of a complex number z is known, the other roots are found by multiplying that root with n^{th} roots of unity.

2. The sum of n^{th} roots of unity is zero.

Example 1.53

Find cubic roots of unity

Solution

$$z = 1, n = 3$$

$$z_k = cis\frac{2k\pi}{3}; \quad k = 0,1,2$$

$$z_0 = \text{cis}\,0 = 1$$

$$z_1 = \text{cis}\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \qquad z_2 = \text{cis}\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Example 1.54

Using cubic roots of unity, find the cubic root of -27, given that -3 is one of the roots.

Solution

We have one of the cubic roots of -27, which is -3.

We have seen that the cubic roots of unity are:

$$\left.\begin{aligned} z_0 &= 1 \\ z_1 &= -\frac{1}{2} + i\frac{\sqrt{3}}{2} \\ z_2 &= -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{aligned}\right\} \Rightarrow \sum_{k=0}^{2} z_k = 0$$

Then, cubic roots of -27 are:

$$z_0 = 1 \times (-3)$$
$$= -3$$

$$z_1 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \times (-3) = \frac{3}{2} - i\frac{3\sqrt{3}}{2}$$

$$z_2 = \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \times (-3) = \frac{3}{2} + i\frac{3\sqrt{3}}{2}$$

Example 1.55

Using 5^{th} roots of unity, find the exact value of $\cos\frac{2\pi}{5}$.

Solution

The 5^{th} roots of unity are given by $z_k = cis\frac{2k\pi}{5},\ k = 0,1,2,3,4.$

$$z_0 = 1$$

$$z_1 = \text{cis}\frac{2\pi}{5} = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$$

$$z_2 = \text{cis}\frac{4\pi}{5} = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$$

$$z_3 = \text{cis}\frac{6\pi}{5} = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$$

$$z_4 = \text{cis}\frac{8\pi}{5} = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}$$

The sum of these roots must be zero, then,

$$1 + \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} + \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} + \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = 0$$

$$1 + \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} + \cos\frac{6\pi}{5} + \cos\frac{8\pi}{5} + i\left(\sin\frac{2\pi}{5} + \sin\frac{4\pi}{5} + \sin\frac{6\pi}{5} + \sin\frac{8\pi}{5}\right) = 0$$

Taking only the real parts, we have;

$$1 + \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} + \cos\frac{6\pi}{5} + \cos\frac{8\pi}{5} = 0 \quad (1)$$

We know that $\cos\alpha = \cos(2\pi - \alpha)$, then,

$$\cos\frac{6\pi}{5} = \cos\left(2\pi - \frac{6\pi}{5}\right) = \cos\left(\frac{10\pi - 6\pi}{5}\right) = \cos\frac{4\pi}{5}$$

$$\cos\frac{8\pi}{5} = \cos\left(2\pi - \frac{8\pi}{5}\right) = \cos\left(\frac{10\pi - 8\pi}{5}\right) = \cos\frac{2\pi}{5}$$

(1) becomes, $1 + \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} + \cos\frac{4\pi}{5} + \cos\frac{2\pi}{5} = 0$ and we have;

$$1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = 0 \quad \Leftrightarrow 1 + 2\cos\frac{2\pi}{5} + 2\cos 2\left(\frac{2\pi}{5}\right) = 0$$

$$\Leftrightarrow 1 + 2\cos\frac{2\pi}{5} + 2\left(2\cos^2\frac{2\pi}{5} - 1\right) = 0, \qquad as\ \cos 2\alpha = 2\cos^2\alpha - 1$$

$$\Leftrightarrow 1 + 2\cos\frac{2\pi}{5} + 4\cos^2\frac{2\pi}{5} - 2 = 0$$

$$\Leftrightarrow 4\cos^2\frac{2\pi}{5} + 2\cos\frac{2\pi}{5} - 1 = 0$$

Let $t = \cos\frac{2\pi}{5}$, we have;

$$4t^2 + 2t - 1 = 0$$

$$\Delta = 4 + 16 = 20$$

$$t_1 = \frac{-2 + \sqrt{20}}{8} = \frac{-2 + 2\sqrt{5}}{8} = \frac{\sqrt{5} - 1}{4}$$

$$t_2 = \frac{-2 - \sqrt{20}}{8} = \frac{-2 - 2\sqrt{5}}{8} = \frac{-\sqrt{5} - 1}{4}$$

But $\frac{2\pi}{5}$ is an angle in the first quadrant. Thus, the cosine of this angle must be positive. Thus, $t = \frac{-\sqrt{5} - 1}{4}$ is to be ignored.

Hence, the exact value of $\cos\frac{2\pi}{5}$ is $\frac{\sqrt{5} - 1}{4}$.

Application activity 1.20

1. Solve the equation $z^4 = i$.
2. Find the five fifth roots of 32.
3. Find five fifth roots of 1.
4. Find four fourth roots of $8+8i\sqrt{3}$.
5. Using 5^{th} roots of unity, find the exact value of $\sin\frac{2\pi}{5}$.

Graphical representation of n^{th} roots of a complex number

Activity 1.21

1. Find five fifth roots of 4.
2. Represent the roots obtained in 1) on Argand diagram.
3. Use a ruler to join the obtained points in 2.

The n roots of a complex number are equally spaced around the circumference of a circle of centre 0 in the complex plane.

If the complex number for which we are computing the n^{th} roots is $z = rcis\theta$, the radius of the circle will be $R = \sqrt[n]{r}$ and the first root z_0 corresponding to $k = 0$ will be at an amplitude of $\varphi = \frac{\theta}{n}$. This root will be followed by the $n-1$ remaining

roots at equal distances apart.

The angular amplitude between each root is $\beta = \frac{2\pi}{n}$.

Now, if $z = 1$, the radius of the circle is 1.

Thus, n^{th} roots of unity are equally spaced around the circumference of a unit circle (circle of centre o and radius 1) in the complex plane.

Example 1.56

Represent graphically the 4^{th} roots of $z = 8\left(1 - i\sqrt{3}\right)$

Solution

$$|z| = 8\sqrt{4} = 16$$

$$\arg z = \arctan \frac{-8\sqrt{3}}{8} = \arctan\left(-\sqrt{3}\right) = -\frac{\pi}{3}$$

The roots are given by:

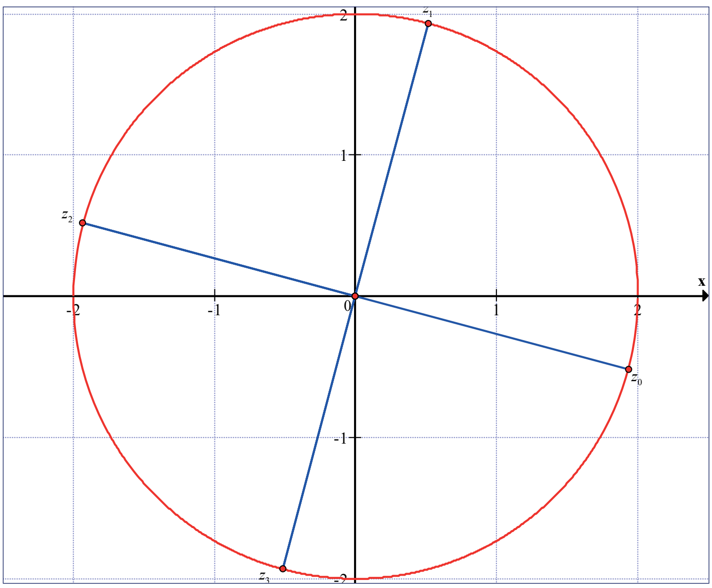
$$z_k = \sqrt[4]{16}\ \text{cis}\left(\frac{-\frac{\pi}{3} + 2k\pi}{4}\right) = 2\,\text{cis}\left(\frac{-\pi + 6k\pi}{12}\right) \text{ where } k = 0, 1, 2, 3.$$

This is;

$$z_0 = 2\,\text{cis}\left(-\frac{\pi}{12}\right) \qquad z_2 = 2\,\text{cis}\left(\frac{11\pi}{12}\right)$$

$$z_1 = 2\,\text{cis}\left(\frac{5\pi}{12}\right) \qquad z_3 = 2\,\text{cis}\left(\frac{17\pi}{12}\right)$$

In this case, the circle will have radius 2.



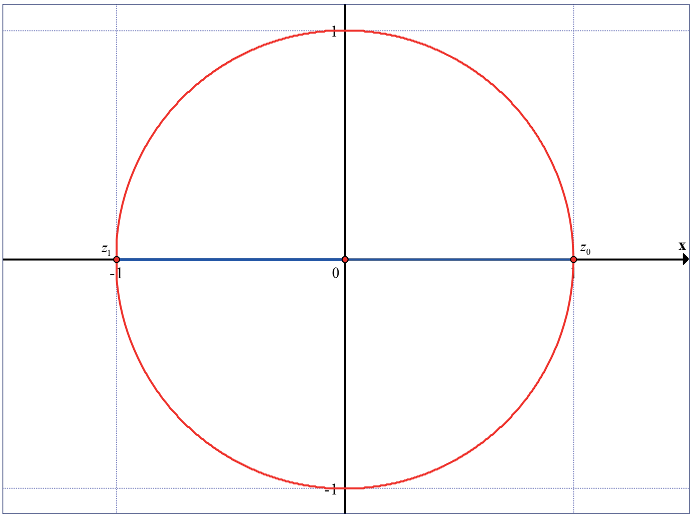
Example 1.57

Represent graphically the n^{th} roots of unity for $n = 2$, $n = 3$ and $n = 4$.

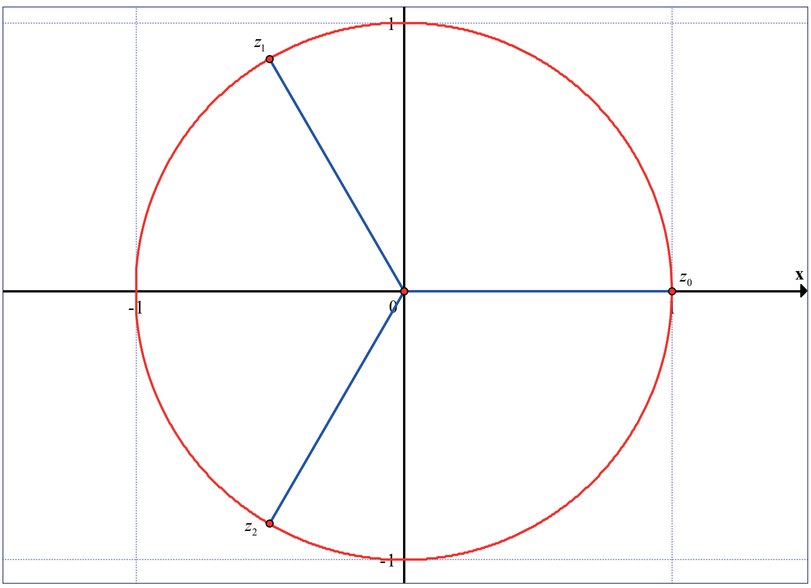
Solution

$$z_k = \text{cis}\left(\frac{2k\pi}{n}\right), \quad k = 0,1,2,3$$

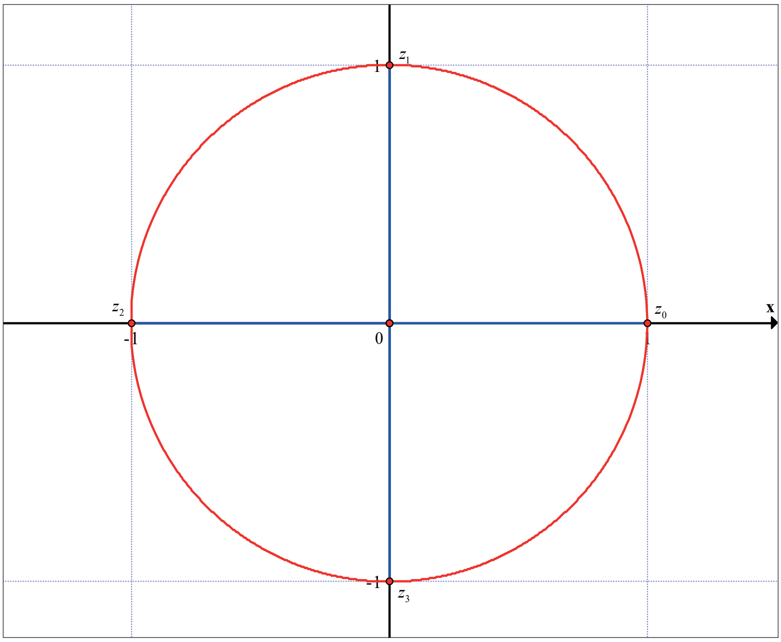
$$n = 2 : z_0 = 1, \quad z_1 = -1$$



$$n = 3 : z_0 = 1, \quad z_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad z_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$



$$n = 4 : z_0 = 1, \quad z_1 = i, \quad z_2 = -1, \quad z_3 = -i$$



We can see that the n^{th} roots of unity for $n > 2$ are the vertices of a regular polygon inscribed in a circle of centre 0 and radius 1.

Application activity 1.21

On Argand diagram, represent:

1. The three cube roots of -27.
2. The four fourth roots of -4.
3. The cube roots of $8i$.
4. The fourth roots of -1.

Construction of regular polygons

A regular polygon is a polygon that is equiangular (all angles are equal in measure) and equilateral (all sides have the same length).

As illustrated in figure 1.10, we call **apothem**, the perpendicular distance from the centre (the interior point) to any side. We can draw a line segment from the centre to one of the vertices. The length of this segment is called the **radius** of the polygon.

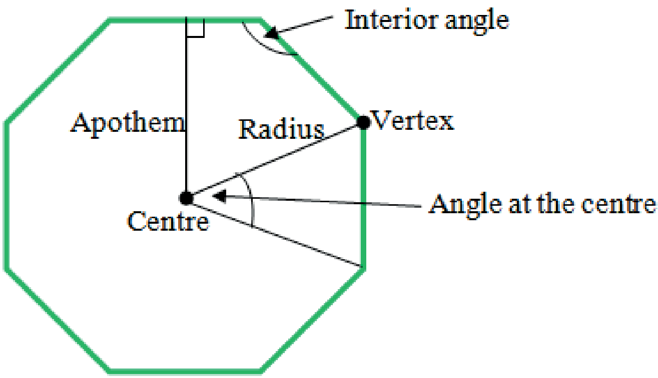


Figure 1.10. *Regular polygon*

Activity 1.22

1. Find cube roots of unity.
2. Represent the roots obtained in 1) on Argand diagram.
3. Using a ruler, join the points obtained in 2).
4. What can you say about the figure obtained in 3)?

Recall that the n^{th} roots of unity are given by:

$$z_k = \text{cis}\frac{2k\pi}{n}, \quad k = 0,1,2,.....,n-1.$$

The n^{th} roots of unity for $n \geq 3$ are the vertices of a regular polygon with n sides inscribed in a circle of centre 0 and radius 1. The vertices of a polygon are the points where its sides intersect. The angle at the centre is given by $\frac{2\pi}{n}$.

To draw a regular polygon with n sides, the essential steps are:

- Start by drawing a unit circle in Argand diagram. The radius and the centre of this circle will be the radius and centre of the regular polygon.
- Around the circle, place the points with affixes $z_k = \text{cis}\frac{2k\pi}{n}, \ k = 0,1,2,.....,n-1$. Those points are the vertices of the polygon.

- Using a ruler, join the obtained points around the circle.
- The obtained figure is the needed regular polygon.

Example 1.58

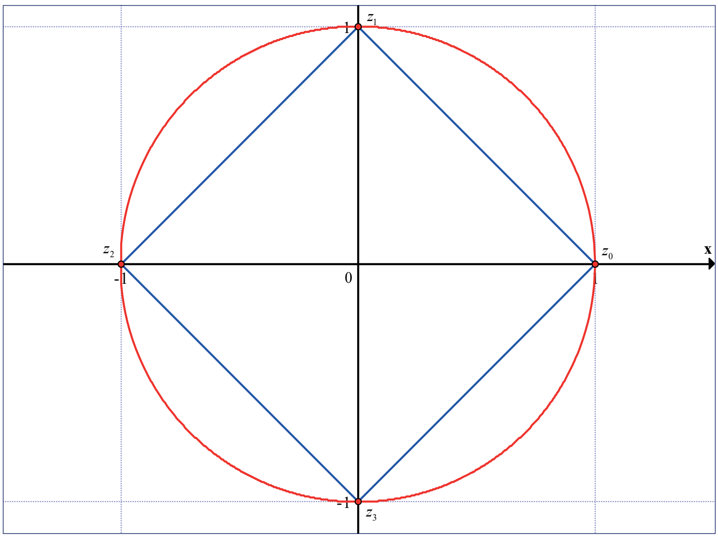
Construct, in Argand diagram, a square.

Solution

A square is a regular polygon with four sides.

We have four vertices: $z_k = \text{cis}\dfrac{2k\pi}{4} = \text{cis}\dfrac{k\pi}{2};\ k = 0,1,2,3$

$z_0 = \text{cis}\,0 = 1,\ \ z_1 = \text{cis}\dfrac{\pi}{2} = i,\ \ z_2 = \text{cis}\,\pi = -1,\ \ z_3 = \text{cis}\dfrac{3\pi}{2} = -i$



Example 1.59

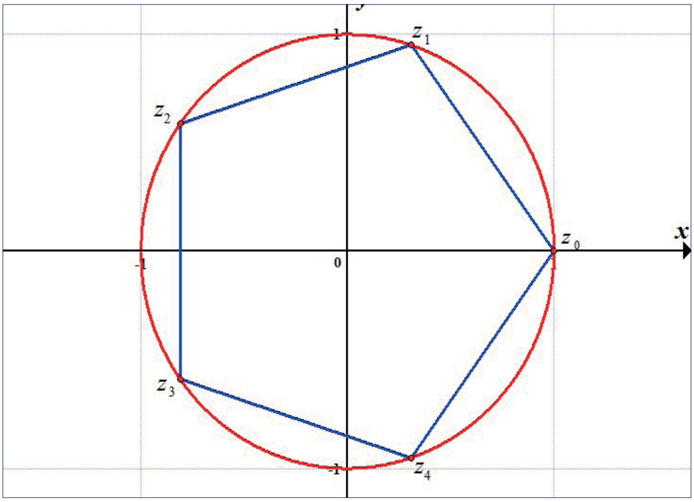
Construct, in Argand diagram, a regular pentagon.

Solution

A regular pentagon has 5 sides.

We have five vertices: $z_k = \text{cis}\dfrac{2k\pi}{5};\ k = 0,1,2,3,4$

$z_0 = \text{cis}\,0 = 1,\ \ z_1 = \text{cis}\dfrac{2\pi}{5},\ \ z_2 = \text{cis}\dfrac{4\pi}{5},\ \ z_3 = \text{cis}\dfrac{6\pi}{5},\ \ z_4 = \text{cis}\dfrac{8\pi}{5}$



Application activity 1.22

In Argand diagram, construct the following polygons:

1. A regular hexagon
2. A regular heptagon
3. A regular octagon
4. A regular nonagon

1.4. Exponential form of a complex number

Activity 1.23

Consider the following infinite series expansions:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \ldots,$$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \ldots$ and $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots$

1. In expansion of e^x, replace x with $i\theta$ and write the new expansion.
2. Rearrange the terms obtained in 1) and use expansions of $\cos x$ and $\sin x$ to find new the expression of e^x in terms of $\cos x$ and $\sin x$.
3. What can you say about the new expression obtained in 2)?

From Activity 1.23, we can write;

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Thus, the exponential form of a complex number z whose modulus is r and argument is θ, is

$$z = re^{i\theta}$$

Example 1.60

Express the complex number $\sqrt{3}+i$ in exponential form.

Solution

$$\left|\sqrt{3}+i\right| = 2,\ \arg\left(\sqrt{3}+i\right) = \frac{\pi}{6}$$

Thus, $\sqrt{3}+i = 2e^{i\frac{\pi}{6}}$

Example 1.61

Express the complex number $-1+i\sqrt{3}$ in exponential form.

Solution

$$\left|-1+i\sqrt{3}\right| = 2,\ \arg\left(-1+i\sqrt{3}\right) = \frac{2\pi}{3}$$

Thus,

$$-1+i\sqrt{3} = 2e^{i\frac{2\pi}{3}}$$

The formulae for product, quotient and power become;

a) $re^{i\theta}r'e^{i\theta'} = rr'e^{i(\theta+\theta')}$

b) $\dfrac{re^{i\theta}}{r'e^{i\theta'}} = \dfrac{r}{r'}e^{i(\theta-\theta')}$

c) $\left(re^{i\theta}\right)^n = r^n e^{in\theta}$

Notice

From exponential form of a complex number, we can find real part and imaginary part as follows:

$$e^{i\theta} = \cos\theta + i\sin\theta \quad (1)$$

$$e^{-i\theta} = \cos\theta - i\sin\theta \quad (2)$$

$(1)+(2)$ gives $e^{i\theta} + e^{-i\theta} = 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$

$(1)-(2)$ gives $e^{i\theta} - e^{-i\theta} = 2i\sin\theta \Rightarrow \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

The formulae

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

are called the **Euler's formulae**.

The Euler's formulae are used to linearise trigonometric expressions. This method is called **linearisation**. We will see this in applications of complex numbers.

Application activity 1.23

Express the following complex numbers in exponential form.

1. i	2. $-1+i\sqrt{3}$	3. $2-2i$	4. $3-i\sqrt{3}$
5. -5	6. $3+3i$	7. $3+4i$	8. $-5-12i$

1.5. Applications

1.5.1. Trigonometric numbers of a multiple of an angle

Activity 1.24

We know that De Moivre's formula is given by

$$(\cos x + i\sin x)^n = \cos nx + i\sin nx \quad (a)$$

1. Use the Newton binomial expansion to expand $(\cos x + i\sin x)^n$ to get expression (b).
2. In relation (a), replace the left hand side with its corresponding expression obtained in 1.
3. Rearrange the terms of relation obtained in (b) to obtain the expression equivalent to $\cos nx$ and another expression equivalent to $\sin nx$. (Recall that two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal).

From Activity 1.24,

$$\cos nx = {}^nc_0\cos^n x - {}^nc_2\cos^{n-2} x\sin^2 x + {}^nc_4\cos^{n-4} x\sin^4 x + ...$$

$$\sin nx = {}^nc_1\cos^{n-1}\sin x - {}^nc_3\cos^{n-3} x\sin^3 x + {}^nc_5\cos^{n-5} x\sin^5 x + ...$$

In general,

$$\cos nx = \sum_{0\le k\le n} {}^nC_k i^k \cos^{n-k} x\sin^k x;\ \textit{for even values of } k$$

$$i\sin nx = \sum_{1\le k\le n} {}^nC_k i^k \cos^{n-k} x\sin^k x;\ \textit{for odd values of } k$$

and ${}^nC_k = \dfrac{n!}{k!(n-k)!}$

Example 1.62

Express $\cos 3x$ and $\sin 3x$ in terms of $\cos x$ and $\sin x$.

Solution

Method 1 (use of De Moivre's formula and binomial expansion)

$(\cos x + i\sin x)^3 = \cos 3x + i\sin 3x$, by De Moivre's formula.

By binomial expansion, we have

$$\begin{aligned}(\cos x + i\sin x)^3 &= \cos^3 x + 3\cos^2 x(i\sin x) + 3\cos x(i\sin x)^2 + (i\sin x)^3 \\ &= \cos^3 x + 3i\cos^2 x\sin x - 3\cos x\sin^2 x - i\sin^3 x \\ &= \cos^3 x - 3\cos x\sin^2 x + i(3\cos^2 x\sin x - \sin^3 x)\end{aligned}$$

Then, $\cos 3x + i\sin 3x = \cos^3 x - 3\cos x\sin^2 x + i(3\cos^2 x\sin x - \sin^3 x)$

Equating parts, we have:

$$\begin{cases}\cos 3x = \cos^3 x - 3\cos x\sin^2 x & (real\ parts) \\ \sin 3x = 3\cos^2 x\sin x - \sin^3 x & (imaginary\ parts)\end{cases}$$

Simplifying the RHS of each expression, we have:

$$\Rightarrow \begin{cases}\cos 3x = \cos^3 x - 3\cos x(1-\cos^2 x) \\ \qquad\quad = \cos^3 x - 3\cos x + 3\cos^3 x \\ \sin 3x = 3(1-\sin^2 x)\sin x - \sin^3 x \\ \qquad\quad = 3\sin x - 3\sin^3 x - \sin^3 x\end{cases} \qquad \left[\text{since } \cos^2 x + \sin^2 x = 1\right]$$

Thus,

$$\begin{cases}\cos 3x = 4\cos^3 x - 3\cos x \\ \sin 3x = 3\sin x - 4\sin^3 x\end{cases}$$

Method 2 (use of the general formulae)

$$\cos nx = \sum_{0\le k\le n} {}^nC_k i^k \cos^{n-k} x\sin^k x;\ \ for\ even\ values\ of\ k$$

$$i\sin nx = \sum_{0\le k\le n} {}^nC_k i^k \cos^{n-k} x\sin^k x;\ \ for\ odd\ values\ of\ k$$

$$\begin{cases}\cos 3x = {}^3C_0 i^0\cos^3 x\sin^0 x + {}^3C_2 i^2\cos^1 x\sin^2 x \\ i\sin 3x = {}^3C_1 i^1\cos^2 x\sin^1 x + {}^3C_3 i^3\cos^0 x\sin^3 x\end{cases}$$

From the above, we obtain;

$$\Rightarrow \begin{cases} \cos 3x = \cos^3 x - 3\cos x \sin^2 x \\ i\sin 3x = 3i\cos^2 x \sin x - i\sin^3 x \end{cases}$$

Simplifying the RHS of each expression, we have:

$$\Rightarrow \begin{cases} \cos 3x = \cos^3 x - 3\cos x\left(1-\cos^2 x\right) \\ i\sin 3x = i\left[3\left(1-\sin^2 x\right)\sin x - \sin^3 x\right] \end{cases}$$

$$\Rightarrow \begin{cases} \cos 3x = \cos^3 x - 3\cos x + 3\cos^3 x \\ i\sin 3x = i\left[3\sin x - 3\sin^3 x - \sin^3 x\right] \end{cases}$$

$$\Rightarrow \begin{cases} \cos 3x = 4\cos^3 x - 3\cos x \\ \sin 3x = 3\sin x - 4\sin^3 x \qquad \textit{as before.} \end{cases}$$

Example 1.63

Express $\tan 4\theta$ in terms of $\tan\theta$.

Solution

$\tan 4\theta = \dfrac{\sin 4\theta}{\cos 4\theta}$, so, expressions for $\sin 4\theta$ and $\cos 4\theta$ in terms of $\sin\theta$ and $\cos\theta$ must be first established as shown below:

$$\begin{cases} \cos 4\theta = {}^4C_0 i^0 \cos^4\theta \sin^0\theta + {}^4C_2 i^2 \cos^2\theta \sin^2\theta + {}^4C_4 i^4 \cos^0\theta \sin^4\theta \\ \text{i}\sin 4\theta = {}^4C_1 i^1 \cos^3\theta \sin^1\theta + {}^4C_3 i^3 \cos^1\theta \sin^3\theta \end{cases}$$

$$\Rightarrow \begin{cases} \cos 4\theta = \cos^4\theta - 6\cos^2\theta \sin^2\theta + \sin^4\theta \\ i\sin 4\theta = 4i\cos^3\theta \sin\theta - 4i\cos\theta \sin^3\theta \end{cases}$$

$$\Rightarrow \begin{cases} \cos 4\theta = \cos^4\theta - 6\cos^2\theta \sin^2\theta + \sin^4\theta \\ \sin 4\theta = 4\cos^3\theta \sin\theta - 4\cos\theta \sin^3\theta \end{cases}$$

Now,

$$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta} = \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$$

Dividing every term in both numerator and denominator by $\cos^4\theta$ gives;

$$= \frac{4\frac{\sin\theta}{\cos\theta} - 4\frac{\sin^3\theta}{\cos^3\theta}}{1 - 6\frac{\sin^2\theta}{\cos^2\theta} + \frac{\sin^4\theta}{\cos^4\theta}} = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

Thus,

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

Application activity 1.24

1. Express $\cos 2x$ in terms of $\cos x$ only.
2. Determine $\sin 2x$ in terms of $\cos x$ and $\sin x$.
3. Express $\tan 3x$ in term of $\cot x$
4. Express $\cot 3x$ in terms of $\cot x$ only.
5. Establish $\tan 5x$ in terms of $\tan x$ only.
6. Evaluate $\cos 6x$ in terms of $\cos x$ only.
7. Determine $\sin 6x$ in terms of $\cos x$ and $\sin x$.

1.5.2. Linearisation of trigonometric expressions (product to sum)

Activity 1.25

Using Euler's formulae, find the sum equivalent to the product $\sin^2 x \cos x$

Recall on page 58 we mentioned that the formulae used in linearisation of trigonometric expressions are Euler's formulae.

Example 1.64

Linearise $2\sin x \cos y$

Solution

$$2\sin x\cos y = \cancel{2}\left(\frac{e^{ix} - e^{-ix}}{2i}\right)\left(\frac{e^{iy} + e^{-iy}}{\cancel{2}}\right)$$

$$=\frac{1}{2i}\left(e^{i(x+y)}+e^{i(x-y)}-e^{i(y-x)}-e^{i(-x-y)}\right)$$

$$=\frac{1}{2i}\left(e^{i(x+y)}+e^{i(x-y)}-e^{-i(x-y)}-e^{-i(x+y)}\right)$$

$$=\frac{1}{2i}\left(e^{i(x+y)}-e^{-i(x+y)}+e^{i(x-y)}-e^{-i(x-y)}\right)$$

$$=\frac{e^{i(x+y)}-e^{-i(x+y)}}{2i}+\frac{e^{i(x-y)}-e^{-i(x-y)}}{2i}$$

$$=\sin(x+y)+\sin(x-y)$$

Example 1.65

Linearise $\sin x\cos^2 x$

Solution

$$\sin x\cos^2 x=\left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{ix}+e^{-ix}}{2}\right)^2 \quad =\left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{2ix}+e^{-2ix}+2}{4}\right)$$

$$=\frac{e^{3ix}+e^{-ix}+2e^{ix}-e^{ix}-e^{-3ix}-2e^{-ix}}{8i}$$

$$=\frac{1}{4}\left(\frac{e^{3ix}-e^{-3ix}}{2i}+\frac{e^{ix}-e^{-ix}}{2i}\right) \quad =\frac{1}{4}(\sin 3x+\sin x)$$

Application activity 1.25

Linearise the following expressions:

1) $\cos x\cos y$
2) $\sin x\sin y$
3) $\sin x\cos x$
4) $\sin^2 x$
5) $\cos^2 x$
6) $\sin^3 x$
7) $\cos^3 x$
8) $\sin^2 x\cos^2 x$

1.5.3. Solving equation $a\cos x + b\sin x = c \quad a,b,c \in \mathbb{R} \quad (a,b \neq 0)$

Activity 1.26

Consider the following equation: $\cos x - \sqrt{3}\sin x = -1$.

Comparing the given equation to the equation $a\cos x + b\sin x = c$;

1. Form the complex number $z = a + bi$.
2. Find the modulus of z, i.e, $|z| = \sqrt{a^2 + b^2}$.
3. Find the principal argument of z, i.e, $\arg(z) = \theta$.
4. Rewrite the given equation in the form $\sqrt{a^2 + b^2}\cos(\theta - x) = c$ and hence solve for x.

To solve the equation $a\cos x + b\sin x = c, \quad a,b,c \in \mathbb{R} \quad (a,b \neq 0)$, we first need the reduction expression for $a\cos x + b\sin x$.

From activity 1.26, to reduce $a\cos x + b\sin x \quad a,b \in \mathbb{R}$, the steps followed are:

1. Form the complex number $z = a + bi$.
2. Find the modulus of z, i.e, $|z| = \sqrt{a^2 + b^2}$.
3. Find the principal argument of z, i.e, $\arg(z) = \theta$.
4. The reduction formula is $a\cos x + b\sin x = \sqrt{a^2 + b^2}\cos(\theta - x)$.

Indeed, considering two complex numbers $z = \cos x + i\sin x$ and $z' = a + bi$ and their presentation on Argand plane is in figure 1.11.

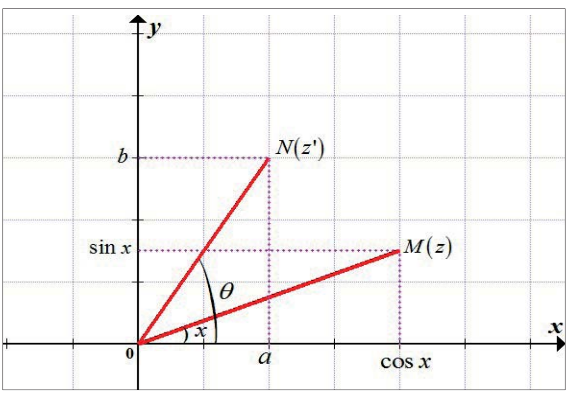


Figure 1.11. *Reduction form of* $a\cos x + b\sin x$

To get the expression equivalent to $a\cos x+b\sin x$, we use dot product expressed in terms of angle $\theta-x$ that is between two vectors $\overrightarrow{OM}=(\cos x,\sin x)$ and $\overrightarrow{ON}=(a,b)$, and then

$$a\cos x+b\sin x=\sqrt{a^2+b^2}\cos(\theta-x).$$

To solve $a\cos x+b\sin x=c$, we use the new equality
$a\cos x+b\sin x=\sqrt{a^2+b^2}\cos(\theta-x)$.

$$\sqrt{a^2+b^2}\cos(\theta-x)=c \Leftrightarrow \cos(\theta-x)=\frac{c}{\sqrt{a^2+b^2}}$$

If

- $\frac{c}{\sqrt{a^2+b^2}}>1$ or $\frac{c}{\sqrt{a^2+b^2}}<-1$, the equation has no solutions.
- $-1\le\frac{c}{\sqrt{a^2+b^2}}\le 1$, the equation has many solutions in set of real numbers.

Example 1.66

Solve, in $\mathbb{R}$, the equation $\cos x+\sqrt{3}\sin x=\sqrt{3}$

Solution

$$z=1+i\sqrt{3},\ |z|=2,\ \arg(z)=\arctan\sqrt{3}=\frac{\pi}{3}$$

$$\Rightarrow \cos x+\sqrt{3}\sin x=2\cos\left(x-\frac{\pi}{3}\right)$$

$$\Rightarrow 2\cos\left(x-\frac{\pi}{3}\right)=\sqrt{3} \quad \Rightarrow \cos\left(x-\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$$

$$\Rightarrow x-\frac{\pi}{3}=\pm\frac{\pi}{6}+2k\pi \quad \Rightarrow x=\begin{cases}\frac{\pi}{2}+2k\pi\\ \frac{\pi}{6}+2k\pi\end{cases},\quad k\in\mathbb{Z}$$

Example 1.67

Solve, in $\mathbb{R}$, the equation $\cos x + \sin x = \sqrt{2}$

Solution

$$z = 1 + i, \quad |z| = \sqrt{2}, \ \arg(z) = \arctan 1 = \frac{\pi}{4}$$

$$\Rightarrow \cos x + \sin x = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) \Rightarrow \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) = \sqrt{2}$$

$$\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = 1 \Rightarrow x - \frac{\pi}{4} = 2k\pi \ \Rightarrow x = \frac{\pi}{4} + 2k\pi, \ k \in \mathbb{Z}$$

Application activity 1.26

Use complex numbers to solve:

1. $\cos x + \sqrt{3}\sin x = \sqrt{3}$
2. $\cos x + \sin x = \sqrt{2}$
3. $\cos x - \sin x = -1$
4. $\sqrt{3}\cos x + \sin x = \sqrt{2}$
5. $2\sin x + \sqrt{3}\cos x = 1 + \sin x$
6. $\sqrt{2}\sec x + \tan x = 1$

1.5.4. Alternating current problems

Activity 1.27

In electrical engineering, the imaginary unit is denoted by j to avoid confusion with i which is generally in use to denote electric current.

If $Z = R + j\omega L + \dfrac{1}{j\omega C}$, express Z in the form $(a + jb)$ when $R = 10, L = 5, C = 0.04$, $\omega = 4$ and $j = \sqrt{-1}$

Where Z denotes **impendance**, R is **resistance**, L is **inductance**, C is **capacitance** and ω indicates the **phasor** for inductance or capacitance.

In electrical engineering, the treatment of resistors, capacitors, and inductors can be unified by introducing imaginary frequency-dependent resistances for the latter two (capacitor and inductor) and combining all the three in a single complex number called the **impedance**. This approach is called

phasor calculus. The imaginary unit is denoted by j to avoid confusion with i which is generally in use to denote electric current.

Since the voltage in an AC circuit is oscillating, it can be represented as

$$\begin{aligned} V &= V_0 e^{jwt} \\ &= V_0 (\cos wt + j \sin wt) \end{aligned}$$

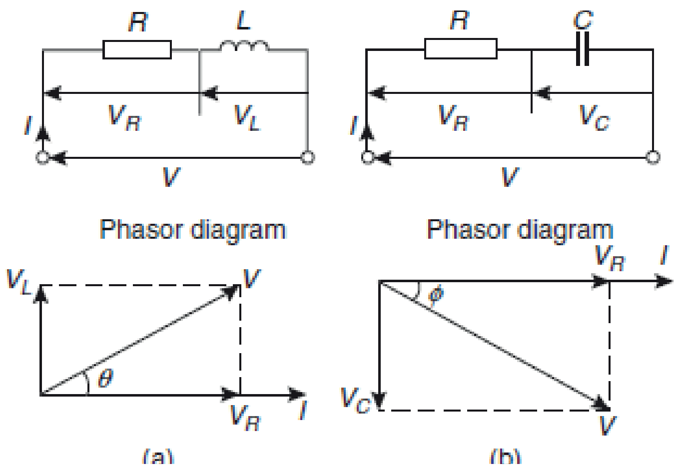
which denotes Impedance, V_o is peak value of impedance and $\omega = 2\pi f$ where f is the frequence of supply.

To obtain the measurable quantity, the real part is taken:

$\mathrm{Re}(V) = V_0 \cos wt$ and is called **Resistance** while imaginary part denotes **Reactance** (inductive or capacitive).

The effect of multiplying a phasor by j is to rotate it in a positive direction (i.e. anticlockwise) on an Argand diagram through 90^0 without altering its length. Similarly, multiplying a phasor by $-j$ rotates phasor in a negative direction (i.e. clockwise) on an Argand diagram through -90^0 without altering its length. These facts are used in alternating current theory since certain quantities in the phasor diagrams lie at 90^0 to each other.

Briefly, **the current, *I*** (cosine function) leads the **applied potential difference** (p.d.), ***V*** (sine function) by one quarter of a cycle i.e. $\frac{\pi}{2}$ radians or 90^0.



For example, in the **Resistance and Inductance** ($R - L$) *series* circuit shown in (a), V_L leads above figure, *I* by 90^0 (i.e. I lags V_L by 90^0) and may be written as jV_L, the vertical axis being regarded as the imaginary axis of an

Argand diagram. Thus, $V_R + jV_L = V$ as $V_R = IR, V_L = IX_L$ (where X_L is the inductive reactance, 2πf L ohms) and V =IZ (where Z is the impedance), then, $R + jX_L = Z$.

Similarly, for the **Resistance** and **Capacitance** ($R - C$) circuit shown in above figure (b), $R - C$ lags I by 90^0 (i.e. I leads V_C by 90^0) and $V_R - jV_C = V$, from which $R - jX_C = Z$ (where X_C is the capacitive reactance $\frac{1}{2\pi fC}$ ohms).

Example 1.68

Determine the resistance and series inductance (or capacitance) for each of the following impedances, assuming a frequency of 50 Hz:

a) $4 + j7\ \ \Omega$ b) $-j20\ \ \Omega$ c) $15cis\left(-60^0\right)\ \ \Omega$

Solution

a) Impedance, $Z = 4 + j7\ \ \Omega$ hence, **Resistance** is 4Ω and **Reactance** $7\ \ \Omega$.

Since the imaginary part is positive, the reactance is inductive, i.e. $X_L = 7\ \ \Omega$

Since $X_L = 2\pi fL$, then **inductance**,

$$L = \frac{X_L}{2\pi f} = \frac{7}{2\pi \times 50} = 0.0223H \text{ or } 22.3mH$$

b) Impedance, $Z = -j20\ \ \Omega$, i.e. $Z = 0 - j20\ \ \Omega$ hence **Resistance** is 0 and **Reactance** 20Ω.

Since the imaginary part is negative, the reactance is capacitive,

i.e. $X_C = 20\ \ \Omega$ and since $X_C = \frac{1}{2\pi fC}$, then,

capacitance,

$$C = \frac{1}{2\pi fX_C} = \frac{1}{2\pi(50)(20)} = \frac{10^6}{2\pi(50)(20)}\mu F = 159.2\mu F$$

c) Impedance,

$$Z = 15cis\left(-60^0\right) = 15\left[\cos\left(-60^0\right) + j\sin\left(-60^0\right)\right] = 7.5 - 7.5j\sqrt{3}$$

Hence, **resistance** is 7.5Ω and **capacitive reactance**, $X_C = 7.5\sqrt{3} = 12.99\Omega$

Since $X_C = \dfrac{1}{2\pi fC}$, then, **capacitance**,

$$C = \frac{1}{2\pi fX_C} = \frac{1}{2\pi(50)(20)} = \frac{10^6}{2\pi(50)(12.99)}\mu F = 245\mu F$$

Example 1.69

An alternating voltage of 240V, 50 Hz is connected across an impedance of $Z = 60 - j100\ \Omega$. Determine:

a) The resistance.

b) The capacitance.

c) The magnitude of the impedance and its phase angle.

d) The current flowing.

Solution

a) Impedance $Z = 60 - j100\ \Omega$.

Hence, **resistance** is 60Ω

b) Capacitive reactance, $X_C = 100\Omega$;

as $X_C = \dfrac{1}{2\pi fC}$, then capacitance,

$$C = \frac{1}{2\pi fX_C} = \frac{1}{2\pi(50)(100)} = \frac{10^6}{2\pi(50)(100)}\mu F = 31.83\mu F$$

c) Magnitude of impedance;

$$|Z| = |60 - j100\ \Omega| = \sqrt{(60)^2 + (-100)^2} = 116.6\Omega$$

Phase angle, $\arg(Z) = \tan^{-1}\left(\dfrac{-100}{60}\right) = -59.04^0$

d) Current flowing; $I = \dfrac{V}{Z} = \dfrac{240cis0^0}{116.6cis(-59.04^0)} = 2.058cis(59.04^0)A$

Application activity 1.27

1. Determine the resistance R and series inductance L (or capacitance C) for each of the following impedances assuming the frequency to be 50 Hz.

   a) $(3+j8)\Omega$ b) $(2-j3)\Omega$

   c) $j14\Omega$ d) $8cis(-60^0)\Omega$

2. Two impedances, $Z_1=(3+j6)\Omega$ and $Z_2=(4-j3)\Omega$ are connected in series to supply a voltage of 120V. Determine the magnitude of the current and its phase angle relative to the voltage.

3. If the two impedances in Problem 2 are connected in parallel, determine the current flowing and its phase relative to the 120V supply voltage.

   Hint: For the n-branch parallel circuit, Impedance Z is given by: $\frac{1}{Z}=\sum_{k=1}^{n}\frac{1}{Z_k}$.

4. A $2.0H$ inductor of resistance 80Ω is connected in series with a 420Ω resistor and a $240V$, $50Hz$ supply. Find;

   a) The current in the circuit.

   b) The phase angle between the applied p.d. and the current.

5. For a transmission line, the characteristic impedance Z_0 and the propagation coefficient γ are given by:

   $$Z_0=\sqrt{\frac{R+j\omega L}{G+j\omega C}}$$

   and

   $$\gamma=\sqrt{(R+j\omega L)(G+j\omega C)}.$$

   Determine, in polar form, Z_0 and γ, given that $R=25\Omega$, $L=5\times10^{-3}H$, $G=80\times10^{-6}S$, $C=0.04\times10^{-6}$ and γ =2000π rad/s.

Unit Summary

1. Concepts of complex numbers

A **complex number** is a number that can be put in the form $a+bi$, where *a* and *b* are real numbers and $i = \sqrt{-1}$.

The set of all complex numbers is denoted by $\mathbb{C}$ and is defined as $\mathbb{C} = \left\{ z = a+bi : a,b \in \mathbb{R} \;\; and \;\; i^2 = -1 \right\}$.

The real number a of the complex number $z = a + bi$ is called the **real part** of z, and the real number b is often called the **imaginary part**. A complex number whose real part is zero is said to be **purely imaginary**, whereas a complex number whose imaginary part is zero is said to be a **real number** or simply **real**.

2. Algebraic form of a complex number

Powers of i: $i^{4k} = 1,\; i^{4k+1} = i,\; i^{4k+2} = -1,\; i^{4k+3} = -i$

$z = (a,b)$ is a **geometric** form of the complex number z and $z = a+bi$ is the **algebraic** (or **standard** or **Cartesian** or **rectangular**) form of the complex number z.

If two complex numbers, say $a+bi$ and $c+di$ are equal, then, both their real and imaginary parts are equal. That is, $a+bi = c+di \;\; \Leftrightarrow a = c \;\; and \;\; b = d$.

The **addition and subtraction** of two complex numbers $a+bi$ and $c+di$ is defined by the formula: $(a+bi) \pm (c+di) = (a \pm c) + (b \pm d)i$.

The **complex conjugate** of the complex number $z = x+yi$, denoted by $\overline{z}$ or z^*, is defined to be $\overline{z} = x - yi$.

The complex number $z = -x - yi$ is the **opposite** of $z = x + yi$, symmetric of z with respect to 0.

The **multiplication** of two complex numbers $a+bi$ and $c+di$ is defined by the formula: $(a+bi)(c+di) = (ac-bd) + (bc+ad)i$

The **inverse** of $z = a+bi$ is given by $\dfrac{1}{z} = z^{-1} = \dfrac{\overline{z}}{a^2+b^2}$

If $z_1 = a+bi \;\; and \;\; z_2 = c+di$ then, $\dfrac{z_1}{z_2} = \dfrac{a+bi}{c+di} = \left(\dfrac{ac+bd}{c^2+d^2}\right) + \left(\dfrac{bc-ad}{c^2+d^2}\right)i$

If a complex number $x + yi$ is a square root of the complex number $a+bi$,

then $\begin{cases} x = \pm\sqrt{\dfrac{1}{2}\left(a+\sqrt{a^2+b^2}\right)} \\ y = \pm\sqrt{\dfrac{1}{2}\left(\sqrt{a^2+b^2}-a\right)} \end{cases}$

Let a, b and c be real numbers ($a \neq 0$), then the equation $az^2+bz+c=0$ has either two real roots, one double real root or two conjugate complex roots.

a) If $\Delta > 0$, there are two distinct real roots:

$$z_1 = \frac{-b+\sqrt{\Delta}}{2a} \text{ and } z_2 = \frac{-b-\sqrt{\Delta}}{2a}.$$

b) If $\Delta = 0$, there is a double real root: $z_1 = z_2 = -\dfrac{b}{2a}$

c) If $\Delta < 0$, there is no real roots. In this case, there are two conjugate complex roots:

$$z_1 = \frac{-b+i\sqrt{-\Delta}}{2a} \text{ and } z_2 = \frac{-b-i\sqrt{-\Delta}}{2a}.$$

Where $\Delta = b^2 - 4ac$

$$z_1 + z_2 = -\frac{b}{a},\ z_1 \cdot z_2 = \frac{c}{a}$$

Every polynomial of positive degree with coefficients in the system of complex numbers has a zero in the system of complex numbers.

Moreover, every such polynomial can be factored linearly in the system of complex numbers.

3. Polar form of a complex number

The **absolute value** (or **modulus** or **magnitude)** of a complex number $z = x+yi$ is $r = |z| = \sqrt{x^2+y^2}$

Principal argument of a complex number $z = x+yi$

$$\arg(z)=\begin{cases}\arctan\frac{y}{x}, & if\ \ x>0\\ \pi+\arctan\frac{y}{x}, & if\ \ x<0, y\geq 0\\ -\pi+\arctan\frac{y}{x}, & if\ \ x<0,\ y<0\\ \frac{\pi}{2}, & if\ \ x=0,\ y>0\\ -\frac{\pi}{2}, & if\ \ x=0,\ y<0\\ Undefined & if\ \ x=0\, and\ y=0\end{cases}$$

Polar (or **modulus-argument**) **form** is $z=r(\cos\theta+i\sin\theta)$ or $z=r\,\text{cis}\,\theta$

Given two complex numbers $z_1=r_1(cos\ \ \theta_1+isin\ \ \theta_1)$ and $z_2=r_2(cos\ \ \theta_2+isin\ \ \theta_2)$, the formulae for multiplication and division are $z_1z_2=r_1r_2(cos\ \ (\theta_1+\theta_2)+isin\ \ (\theta_1+\theta_2))$ and $\frac{z_1}{z_2}=\frac{r_1}{r_2}(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2))$ respectively.

Power of a complex number z is given by

$$z^n=(r(\cos\theta+i\sin\theta))^n=r^n(\cos n\theta+i\sin n\theta);\ n\in\mathbb{Z}_0$$

De Moivre's theorem: $(\cos\theta+i\sin\theta)^n=(\cos n\theta+i\sin n\theta)$

If $(z_k)^n=z$ for $z=rcis\theta$, then

$$z_k=\sqrt[n]{r}\ cis\left(\frac{\theta+2k\pi}{n}\right)\quad k=0,1,2,3,......,n-1$$

To draw a regular polygon with n sides, the steps followed are:

» Start by drawing a unit circle in Argand diagram. The radius and the centre of this circle will be the radius and centre of the regular polygon.

» Around the circle, place the points with affixes
$z_k=\text{cis}\frac{2k\pi}{n},\ k=0,1,2,...,n-1$.

» Those points are the vertices of the polygon.

» Using a ruler, join the obtained points around the circle.

» The obtained figure is the needed regular polygon.

4. Exponential form of a complex number

The exponential form of a complex number z whose modulus is r and argument is θ, is $z = re^{i\theta}$.

Euler's formulae (these formulae are used to linearise trigonometric expressions):

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

5. Applications

» Formulae for trigonometric number of a multiple of an angle

$$\cos nx = \sum_{0\le k\le n} {}^nC_k i^k \cos^{n-k} x \sin^k x, \textit{ with even } k \textit{ values}$$

$$i\sin nx = \sum_{0\le k\le n} {}^nC_k i^k \cos^{n-k} x \sin^k x, \textit{ with odd } k \textit{ values } {}^nC_k = \frac{n!}{k!(n-k)!}$$

» To solve the equation $a\cos x + b\sin x = c$, solve the equation

$$\cos(x-\theta) = \frac{c}{\sqrt{a^2+b^2}}, \quad \theta = \arg(a+bi)$$

» Alternating current

Resistance and Capacitance (R-C)

Let a p.d. V be applied across a resistance R and a capacitance C in series. The same current I flows through each component and so the reference vector will be that representing I. The p.d. V_R across R is in phase with I, and V_C, that across C, lags on current I by 90^0.

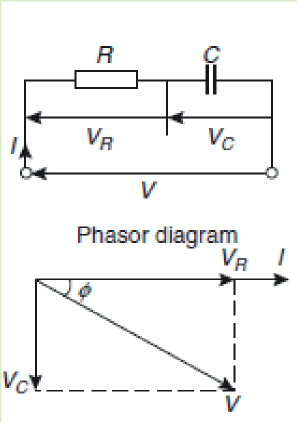


Figure showing Resistance and Capacitance in series

Vector sum of V_R and V_C is called **Impedance** and equals the applied p.d. V;

$Z = V_R + jV_C$ where V_R and V_C are known as **resistance** and **reactance** respectively.

But $V_R = IR$ and $V_C = IX_C$ where X_C is the **capacitive reactance** of C and equals $\frac{1}{\omega C}$.

Resistance and inductance (R-L)

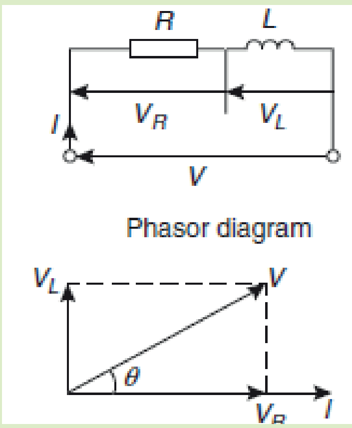


Figure showing Resistance and Inductance in series

The analysis is similar but here, the p.d. V_L across L leads on current I and the p.d. V_R across R is again in phase with I.

$Z = V_R + jV_L$ where V_R and V_L are known as **resistance** and **reactance** respectively.

But $V_R = IR$ and $V_L = IX_L$ where X_L is the **inductive reactance** of L and equals ωL

or $\omega = 2\pi f$.

For the ***n–branch parallel circuit***, Impedance Z is given by: $\frac{1}{Z} = \sum_{k=1}^{n} \frac{1}{Z_k}$

End of unit assessment

1. For the complex numbers $z = 3 - i$ and $w = 1 + 2i$, evaluate;

   a) $2z - 3w$ b) zw

   c) $z^2 w(\bar{z})^2$ d) $\dfrac{z}{w}$

2. Solve the following equations in $a + bi$ form:

   a) $z^2 + 4 = 0$ b) $z^2 + z + 1 = 0$

   c) $z^2 + 6z + 11 = 0$ d) $z^3 - 1 = 0$

3. Plot the following complex numbers on the Argand plane and express them into polar form.

   a) 1 b) i c) $-3i$

   d) $1 - i$ e) $2 + i$ f) $-3 - 2i$

   g) $-3 + 2i$

4. Convert into Cartesian form;

   a) $2 \operatorname{cis} 0$ b) $3 \operatorname{cis} \pi$

   c) $\operatorname{cis} \dfrac{\pi}{2}$ d) $3 \operatorname{cis} \dfrac{3\pi}{4}$

5. By conversion to polar form and use of De Moivres' theorem, evaluate;

   a) i^7 b) $(1+i)^5$ c) $\left(\sqrt{3} - i\right)^{-4}$

6. Find in $a + bi$ form and plot on Argand diagram;

   a) the three values of $(i)^{\frac{1}{3}}$.

   b) the four values of $(1+i)^{\frac{1}{4}}$.

7. If $z_1 = \operatorname{cis}\left(\dfrac{\pi}{4}\right)$ and $z_2 = \operatorname{cis}\left(-\dfrac{\pi}{3}\right)$, evaluate $\dfrac{z_1}{z_2}$ in polar and Cartesian form.

   Deduce the exact value of $\cos\left(\dfrac{7\pi}{12}\right)$ and $\sin\left(\dfrac{7\pi}{12}\right)$.

8. Find the modulus of $e^{i\theta}$

9. Show that $\left(e^{i\theta}\right)^{-1} = e^{-i\theta}$

10. Show that the sum of n^{th} roots of unity is zero.

11. Using 10^{th} roots of unity, find the exact value of $\cos\frac{\pi}{5}$.

12. Determine the set of complex numbers, z, such that $z = z^2 + 3z + 4$ is a real number.

13. Solve the equation $z^3 + 3z^2 + 4z + 12 = 0$

14. Simplify $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{2001}$

15. Consider the points A and B with affixes $1-i$ and $1+i$ respectively. Let θ be a real number of interval $]0, 2\pi[$,

    distinct from π, and r rotation of centre 0 through angle $\frac{\pi}{2}$.

    Note: if M is the point with affix $1+ie^{i\theta}$, then M' is the image of M by rotation r.

    a) Show that M is a point circle of diameter AB.

    b) Show that the points B, M and M' are collinear.

16. Find the values of number x for which $\left[10 - x + i(2+x)\right](x-i)$ is real.

17. In each case, determine the set of points of M of complex plane, with affix z such that;

    a) $|z-2| = |z+1|$ b) $z-2i| = |z+2|$ c) $|z-1+3i| = 2$

    d) $z + \overline{z} + z(\overline{z}) = 0$ e) $\operatorname{Im}\left(z^2\right) = 2$ f) $\operatorname{Re}\left(z^2\right) = 0$

18. Determine two complex numbers such that their sum is 2 and their product is 9.

19. Determine complex number(s) z different from zero such that z^2 and z^6 are conjugates.

20. Determine real part, imaginary part, modulus and argument of

$$z=\frac{1-e^{i\frac{\pi}{3}}}{1+e^{i\frac{\pi}{3}}}$$

21. Determine the modulus and argument of

$$z=\frac{1+\cos\theta+i\sin\theta}{1-\cos\theta+i\sin\theta}, \qquad \theta\neq 2k\pi$$

22. Determine complex number(s) z such that $z-i$, $iz-i$ and $z-iz$ have the same modulus.

23. Determine complex number(s) z such that $\left|\frac{z-4}{z-8}\right|=1$

24. In complex plane, the points A, B and C are with affixes $z_1=1+2i$, $z_2=4-2i$, and $z_3=1-6i$ respectively. Show that the triangle ABC is isosceles triangle.

25. a) Determine (i) $\left(\frac{\sqrt{3}-i}{1+i\sqrt{3}}\right)^9$ (ii) $\left(\frac{\sqrt{3}+i}{\sqrt{3}-i}+\frac{\sqrt{3}-i}{\sqrt{3}+i}-2\right)^{30}$

    b) How can you choose a natural number n different from zero such that the number $\left(\sqrt{3}-i\right)^n$ is

    (i) real?

    (ii) pure imaginary?

    b) Show that $(1+i)^n+(1-i)^n=2^{\frac{n+2}{2}}\cos\frac{n\pi}{4}$

26. Consider the equation $z^3+2z^2+2z+1=0$

    a) Show that -1 is a root of equation.

    b) Determine real numbers a, b and c such that $z^3+2z^2+2z+1=(z+1)\left(az^2+bz+c\right)$

    c) Solve the equation E in $\mathbb{C}$.

27. Consider the four points A, B, C and D, on a complex plane with affixes $2-3i$, $\frac{1}{2}$, $1+4i$ and $4+2i$ respectively.

    a) Plot these points on complex plane.

b) Calculate the affixes of vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$.

c) Determine the affix of point E such that $ABCE$ is a parallelogram.

28. Given two complex numbers $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$, ,

a) Write $\frac{Z_1}{Z_2}$ in algebraic and polar forms.

b) Deduce the exact values of $Cos\frac{5\pi}{12}$ and $Sin\frac{5\pi}{12}$.

c) What is the lowest positive value of integer n such that $\left(\frac{z_1}{z_2}\right)^n$ is real?

29. Determine the magnitude and direction of the resultant of the three coplanar forces given below, when they act at a point.

Force A , 10N acting at 450 from the positive horizontal axis.

Force B , 8N acting at 120 from the positive horizontal axis.

Force C , $15N$ acts at 210 from the positive horizontal axis.

30. Determine, using complex numbers, the magnitude and direction of the resultant of the coplanar forces given below, which are acting at a point.

Force A , $5N$ acting horizontally, Force B , $9N$ acting at an angle of 135^0 to force A, Force C , 12N acting at an angle of 240^0 to force A.

31. A delta-connected impedance Z_A is given by:

$$Z_A = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{z_2}$$

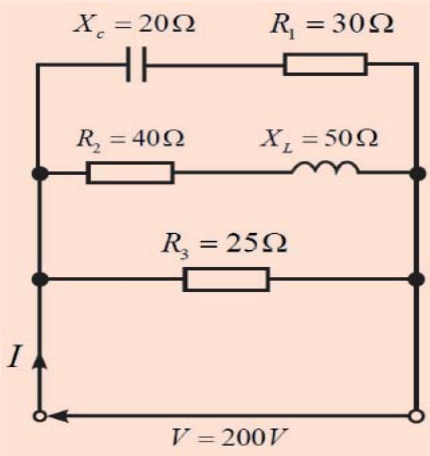
Determine Z_A in both Cartesian and polar form given $Z_1 = (10 + j0)\Omega,\ Z_2 = (0 - j10)\Omega$ and $Z_3 = (10 + j10)\Omega$.

32. In the hydrogen atom, the angular momentum, p, of the de Broglie wave is given by: $p\psi = \left(\frac{jh}{2\pi}\right)(\pm jm\psi)$

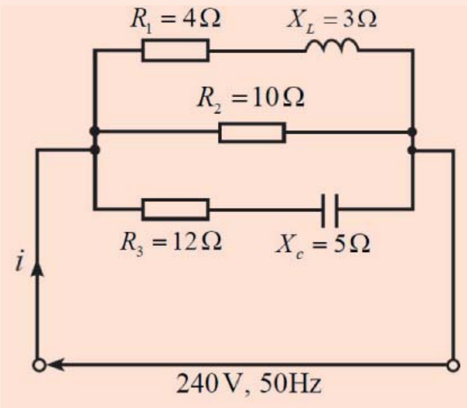
Determine an expression for p.

33. A series circuit consists of a 12Ω resistor, a coil of inductance $0.1H$ and a capacitance of $t160\mu F$ Calculate the current flowing and its phase relative to the supply voltage of $240V, 50Hz$

34. For the circuit shown in the figure below, determine the current *I* flowing and its phase relative to the applied voltage .



35. For the parallel circuit shown in the figure below, determine the value of current *I*, and its phase relative to the 240V supply, using complex numbers.



Unit 2

Logarithmic and Exponential Functions

Introductory activity

Let's look at the following problem: "*The population P of a city increases according to the formula $P = 500e^{at}$ where t is in years and $t = 0$ corresponds to 1980. In 1990, the population was 10000. Discuss how to find the value of the constant a and approximate your answer to 3 decimal places.*"

You must have met many different kinds of functions. You know that each one can be used to model some kind of situation in the real world. Exponential and logarithmic functions are no exception! Much of the power of logarithms is their usefulness in solving exponential equations. Some examples of this include sound (decibel measures), population growth, earthquakes (Richter scale), the brightness of stars, and in chemistry we have the (pH balance, which is a measure of acidity and alkalinity).

Objectives

By the end of this unit, a student will be able to:

- Find the domain of a given logarithmic or exponential function.
- Evaluate the limit of a given logarithmic or exponential function.
- Differentiate a given logarithmic or exponential function.
- Find relative asymptotes of a given logarithmic or exponential function.
- Apply logarithmic or exponential function in real life problems.

From this problem, if $t = 0$ corresponds to 1980, then 1990 corresponds to $t = 10$ and this gives the following equation: $500e^{10a} = 1000$ or $e^{10a} = 2$. The calculation of the value of a leads to the introduction of logarithms. The question is how can logarithmic function be applied. In this unit, you will see how you can solve such kind of problem.

2.1. Logarithmic functions

2.1.1. Natural logarithm

Domain and range of natural logarithmic functions

Activity 2.1

Use calculator to complete the following tables:

x	$\ln x$
-0.8	
-0.6	
-0.4	
-0.2	
0	

x	$\ln x$
0.2	
0.4	
0.6	
0.8	
1	

x	$\ln x$
1.5	
2	
2.5	
3	
3.5	

1. Using the tables, give your observation for
   (i) negative x values and zero.
   (ii) x values between 0 and 1.
   (iii) x values greater than 1.
2. Plot a graph of $y = \ln x$ for $x > 0$.

The **Natural logarithm** of x is denoted $\ln x$ or $\log_e x$.

From Activity 2.1,

$\ln x$ is defined on positive real numbers, $]0,+\infty[$ and its range is all real numbers.

Particularly,

- If $x = 1$, then $\ln x = 0$. That is, $\ln 1 = 0$
- If $x > 1$, then $\ln x > \ln 1$ or $\ln x > 0$
- If $0 < x < 1$, then $\ln x < \ln 1$ or $\ln x < 0$

It means that: $\forall x \in]1,+\infty[$, $\ln x > 0$ and $\forall x \in]0,1[$, $\ln x < 0$

Properties

$\forall x, y \in]0,+\infty[$

a) $\ln xy = \ln x + \ln y$ b) $\ln \frac{1}{y} = -\ln y$

c) $\ln \frac{x}{y} = \ln x - \ln y$ d) $\ln x^r = r \ln x$

Therefore, the range of $f(x) = \ln x$ is $]-\infty, +\infty[$ or $\mathbb{R}$.

Notice

The number e:

The equation $\ln x = 1$ has, in interval $]0, +\infty[$, a unique solution, a rational number $2.718281828459045235360....$. This number is denoted by e.

Thus, $e = 2.718281828459045235360....$

Hence, if $\ln x = 1$ then $x = e$.

The number e is defined to be the limit of the sequence $\left(1+\frac{1}{x}\right)^x$ as x tends to $+\infty$. That is, $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$ or $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$ (it will be proved later).

Example 2.1

Find the domain of definition of $f(x) = \ln(2x-3)$

Solution

Condition: $2x-3>0$

$$2x-3>0 \Leftrightarrow x > \frac{3}{2}$$

Thus, $Domf = \left]\frac{3}{2}, +\infty\right[$

Example 2.2

Find the domain of definition of $f(x) = \ln(x+3)(x+2)$

Solution

Condition: $(x+3)(x+2)>0$

$(x+3)(x+2)>0$ if $x\in\left]-\infty,-3\right[\cup\left]-2,+\infty\right[$ (sign table can be used)

Thus, $Domf=\left]-\infty,-3\right[\cup\left]-2,+\infty\right[$

Application activity 2.1

1. Find the domain of definition of:

   a) $f(x)=\ln\frac{1}{x}$ b) $f(x)=\ln\left(4x-x^2\right)$

   c) $f(x)=\ln\left|4x-x^2\right|$ d) $f(x)=\ln(3-x)+\ln|4-x|$

2. The decibel gain n of an amplifier is given by: $n=10\log\frac{P_2}{P_1}$ where P_1 is the power input and P_2 is the power output. Find the power gain $\frac{P_2}{P_1}$ when n=25 decibels.

2.1.2. Limit and asymptotes for natural logarithmic functions

Activity 2.2

Let $y=\ln x$

1. From the domain of $y=\ln x$, does the $\lim\limits_{x\to 0^-}\ln x$ exist? If NO explain, if YES give its value.
2. If x takes on values closer to 0 from the right, what can you conclude about the values of $\ln x$. Deduce $\lim\limits_{x\to 0^+}\ln x$. Deduce asymptote, if any.
3. Given that x takes on values of the form $10^n\ (n\in\mathbb{N})$, $\ln 10^n=n\ln 10\approx 2.30n$ and let n take on values 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, …, what can you conclude about the values of $\ln x$. Deduce $\lim\limits_{x\to+\infty}\ln x$ and asymptote, if any.

From Activity 2.2,

$\lim_{x\to+\infty} \ln x = +\infty$ and $\lim_{x\to 0^+} \ln x = -\infty$

From $\lim_{x\to+\infty} \ln x = +\infty$, we deduce that there is no horizontal asymptote.

Remember that $\ln\frac{1}{x} = -\ln x$ or $\ln x = -\ln\frac{1}{x}$

$$\lim_{x\to 0^+} \ln x = \lim_{x\to 0^+}\left(-\ln\frac{1}{x}\right) \qquad = -\ln\left(\lim_{x\to 0^+}\frac{1}{x}\right) \qquad\qquad = -\infty$$

$$\left[\text{since } \lim_{x\to 0^+}\frac{1}{x} = +\infty\right]$$

Then, $\lim_{x\to 0^+} \ln x = -\infty$

From this limit, we deduce that there exists a vertical asymptote with equation $VA \equiv x = 0$.

Keep in mind that $\lim_{x\to 0^-} \ln x$ does not exist because the left of 0 is not included in the domain.

Example 2.3

Evaluate $\lim_{x\to+\infty}\frac{\ln x}{x}$

Solution

$$\lim_{x\to+\infty}\frac{\ln x}{x} = \frac{+\infty}{+\infty} \qquad [\text{indeterminate form}]$$

$$= \lim_{x\to+\infty}\frac{\frac{1}{x}}{1} \qquad \left[\text{from Hôpital rule, later we will see how } (\ln x)' = \frac{1}{x}\right]$$

$$= \lim_{x\to+\infty}\frac{1}{x} = 0$$

Thus, $\lim_{x\to+\infty}\frac{\ln x}{x} = 0$

Example 2.4

Evaluate $\lim_{x\to 0^+} x\ln x$

Solution

$$\lim_{x\to 0^+} x\ln x = 0(-\infty)$$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \frac{-\infty}{+\infty} \quad [\text{indeterminate form}]$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \quad \left[\text{from Hôpital rule, later we will see how } (\ln x)' = \frac{1}{x}\right]$$

$$= \lim_{x\to 0^+} (-x) = 0$$

Hence, $\lim_{x\to 0^+} x\ln x = 0$

Application activity 2.2

Evaluate the following limits:

1. $\lim_{x\to 0^+} \frac{1+2\ln x}{x}$
2. $\lim_{x\to +\infty} \frac{1+2\ln x}{x}$
3. $\lim_{x\to -\infty} \ln\left(x^2-4x+1\right)$
4. $\lim_{x\to +\infty} \ln\left(x^2-4x+1\right)$

Derivative of natural logarithmic functions

Activity 2.3

1. Using definition of derivative, find the derivative of $\ln x$.
2. Consider u a differentiable function of x. Use result in 1) and rule of differentiating composite functions to find the derivative of $\ln u$. Refer to the fact

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}, \quad \lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$$

$$(f\circ g)'=f'(g)g'$$

Form Activity 2.3,

$$(\ln x)'=\frac{1}{x}$$

Also, if u is a differentiable function of x then,

$$(\ln u)'=\frac{u'}{u}$$

Example 2.5

Find the derivative of $f(x)=\ln\sqrt{x^2+1}$

Solution

$$f'(x)=\left(\ln\sqrt{x^2+1}\right)' = \frac{\left(\sqrt{x^2+1}\right)'}{\sqrt{x^2+1}}$$

$$=\frac{\dfrac{2x}{2\sqrt{x^2+1}}}{\sqrt{x^2+1}} = \frac{x}{\left(\sqrt{x^2+1}\right)^2} = \frac{x}{x^2+1}$$

Thus, $f'(x)=\dfrac{x}{x^2+1}$

Example 2.6

Differentiate the function $g(x)=\ln(x+\cos x)$

Solution

$$g'(x)=\left[\ln(x+\cos x)\right]' = \frac{(x+\cos x)'}{x+\cos x} = \frac{1-\sin x}{x+\cos x}$$

Thus, $g'(x)=\dfrac{1-\sin x}{x+\cos x}$

Application activity 2.3

Find derivative of the following functions:

1. $f(x)=(\ln x)^2$
2. $g(x)=\ln(\tan x)$
3. $h(x)=\ln\sqrt{x^2+1}$
4. $k(x)=\ln\dfrac{1-x}{1+x}$
5. $f(x)=\dfrac{\ln(\sin x)}{x}$
6. $g(x)=\ln x+\ln(\cos x)$
7. $h(x)=\tan x-\dfrac{\ln x}{3x}$
8. $k(x)=\dfrac{\ln\left(\sqrt{x+1}\right)}{x+1}$

Variation and curve sketching of natural logarithmic functions

Activity 2.4

Let $f(x)=\ln x$

1. From the domain of definition of $f(x)$, evaluate limits at the boundaries of the domain of $f(x)$ and hence deduce relative asymptotes, if any.
2. Determine the first derivative and variation of $f(x)$. Deduce the extrema, if any.
3. Determine the second derivative and concavity of $f(x)$. Deduce the inflection points, if any.

4. Complete the following table

x	0 ...	1	...	e	... $+\infty$
Sign of $f'(x)$					
Sign of $f''(x)$					
Variation of $f(x)$					
Concavity of $f(x)$					

5. Find the intersection of $f(x)$ with axes of co-ordinates.
6. Find additional points and hence sketch the curve of $f(x)$.

From activity 2.4, the curve of the function $f(x)=\ln x$ is given in figure 2.1.

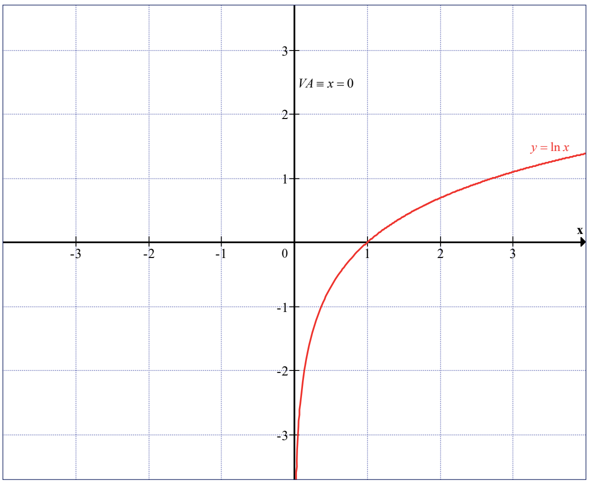


Figure 2.1. *Natural logarithmic function*

Example 2.7

Given the function $f(x)=\dfrac{1+2\ln x}{x}$. Find relative asymptotes (if any), study the variation, concavity and sketch the curve.

Solution

Asymptotes

First, we need domain of definition:

Condition: $x>0 \Rightarrow Domf=\mathbb{R}_0^+$

$$\lim_{x\to 0^+} f(x) = \frac{-\infty}{0} = -\infty \;\Rightarrow x = 0$$ is a vertical asymptote

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{1+2\ln x}{x} = \frac{\infty}{\infty} \quad IF$$

$$= \lim_{x\to+\infty} \frac{\frac{2}{x}}{1} \qquad [\text{Hôpital rule}]$$

$$= 0$$

$\Rightarrow y = 0$ is horizontal asymptote.

Since there is horizontal asymptote for $x \to +\infty$, there is no oblique asymptote for $x \to +\infty$.

Variation

First derivative

$$f'(x) = \frac{\frac{2}{x}x - (1+2\ln x)}{x^2} = \frac{1-2\ln x}{x^2} \text{ with } x > 0$$

Roots of first derivative

$$f'(x) = 0 \Leftrightarrow \frac{1-2\ln x}{x^2} = 0$$

$$\Rightarrow 1-2\ln x = 0 \;\Leftrightarrow 1-\ln x^2 = 0 \;\Leftrightarrow \ln x^2 = 1 \;\Leftrightarrow x^2 = e \Rightarrow x = \pm\sqrt{e}$$

As $x > 0$, $x = -\sqrt{e}$ (is to be rejected).

Thus root of $f'(x) = 0$ is $x = \sqrt{e}$

0	0		$\sqrt{e}$		$+\infty$
$f'(x)$		+	0	−	
$f(x)$	$-\infty$	$\nearrow$	$\frac{2}{\sqrt{e}}$	$\searrow$	0

For $x \in \left]0, \sqrt{e}\right[$, $f(x)$ increases while for $x \in \left]\sqrt{e}, +\infty\right[$, $f(x)$ decreases.

Concavity

Second derivative

$$f''(x)=\left[\frac{1-2\ln x}{x^2}\right]'=\frac{(1-2\ln x)'x^2-2x(1-2\ln x)}{x^4}$$

$$=\frac{-2x-2x(1-2\ln x)}{x^4}=\frac{4x\ln x-4x}{x^4}=\frac{4\ln x-4}{x^3}$$

Roots of second derivative

$$f''(x)=0\Leftrightarrow=\frac{4\ln x-4}{x^3}=0$$

$$\Rightarrow 4\ln x-4=0\Leftrightarrow \ln x-1=0\Rightarrow x=e$$

The root of $f''(x)=0$ is $x=e$

x	0		e		$+\infty$
$f''(x)$		-	0	+	
$f(x)$		∩	$\frac{3}{e}$	∪	

Variation table

x	0		$\sqrt{e}$		e		$+\infty$
$f'(x)$		+	0	−			
$f''(x)$			−		0	+	
$f(x)$	$-\infty$	↗	$\frac{2}{\sqrt{e}}$	↘	$\frac{3}{e}$	↘	0

Curve

Intersection with coordinates axes:

Intersection with $x-axis$;

$$f(x)=0\Leftrightarrow 1+2\ln x=0$$

$$\Leftrightarrow \ln x = -\frac{1}{2} \Rightarrow x = e^{-\frac{1}{2}}$$

Thus, $f(x) \cap ox = \left\{ \left(e^{-\frac{1}{2}}, 0 \right) \right\}$

Intersection with $y-axis$;

$$f(0) = \frac{1 + 2\ln 0}{0} \quad \textit{which is impossible}$$

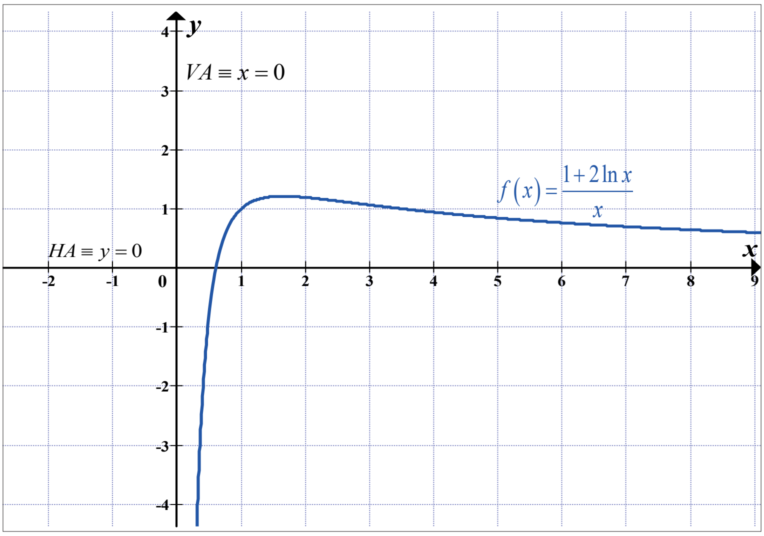
Thus, no intersection with $y-axis$.

Additional points:

x	0.1	0.4	0.7	1.0	1.3	1.6	1.9	2.2	2.5	2.8
$f(x)$	-36.05	-2.08	0.40	1.0	1.17	1.21	1.20	1.17	1.13	1.09

x	3.1	3.4	3.7	4.0	4.3	4.6	4.9	5.2	5.5	5.8
$f(x)$	1.05	1.01	0.97	0.94	0.91	0.88	0.85	0.82	0.80	0.77

Thus, the sketch becomes



Application activity 2.4

For each of the following functions, find relative asymptotes (if any), study the variation and concavity of the function and hence sketch the curve;

1. $f(x) = \ln(x^2)$
2. $g(x) = \ln(x+1)$
3. $h(x) = \dfrac{\ln x}{x}$
4. $k(x) = \ln(x^2 - 3x + 2)$

2.1.3. Logarithmic function with any base

Domain and range of logarithmic function with any base

Activity 2.5

For each of the following functions, determine the domain of definition and range.

1. $f(x) = \dfrac{\ln x}{\ln 2}$
2. $g(x) = \dfrac{\ln x}{\ln \frac{1}{2}}$

We call logarithm of a **real number** x with **base** a the number denoted $\log_a x$, defined by $\log_a x = \dfrac{\ln x}{\ln a}$, $x \in \mathbb{R}_0^+$, $a \in \mathbb{R}_0^+ \setminus \{1\}$
$\forall x \in \mathbb{R}_0^+, \log_a x = y \Leftrightarrow x = a^y$

Special cases:

- If $a = 10$, we simply write $\log x$ and we call it **decimal logarithm**.
- If $a = e$, we have $\log_e x = \dfrac{\ln x}{\ln e} = \ln x$ and this is a natural logarithm.

Note that, $\log_a 1 = 0$, $\log_a a = 1$, $a^{\log_a x} = x$

Properties

$\forall x, y \in]0, +\infty[, a \in]0, +\infty[\setminus \{1\}$:

a) $\log_a xy = \log_a x + \log_a y$ b) $\log_a \dfrac{1}{y} = -\log_a y$

c) $\log_a \dfrac{x}{y} = \log_a x - \log_a y$ d) $\log_a x^r = r \log_a x$

Notice

Recall that $\ln a < 0 \ for \ ]0,1[$ and $\ln a > 0 \ for \ ]1,+\infty[$ .Thus,

$f(x) = \log_a x$ is increasing in $]1,+\infty[$ and $f(x) = \log_a x$ is decreasing in $]0,1[$;

- If $x \in]0,1[, \log_a x < \log_a y \Leftrightarrow x > y$
- If $x \in]1,+\infty[, \log_a x < \log_a y \Leftrightarrow x < y$
- $\log_a x = \log_a y \Leftrightarrow x = y$.

Example 2.8

Example 2.9

Find the domain of $f(x) = \log_3(1-x) + \log_2 x$

Solution

Conditions: $1-x > 0$ and $x > 0$

$1-x > 0 \Rightarrow x < 1$

Domain is the intersection of $x < 1$ and $x > 0$

Thus, $Domf =]0,1[$

Example 2.10

Find the domain of $f(x) = \log_2\left(\frac{\sqrt{x}}{x+1}\right)$

Solution

Conditions: $x > 0$, $x+1 \neq 0$ and $\frac{\sqrt{x}}{x+1} > 0$

$x+1 \neq 0 \Rightarrow x \neq -1$

x	$-\infty$	-1		0	$+\infty$
$\sqrt{x}$				0	+
$x+1$	$-$	0	+	+	+

$\frac{\sqrt{x}}{x+1}$		0	+

Thus, $Domf =]0,+\infty[$

Example 2.11

Let $y=\log_b x$, express y as a function of $\log_a$

Solution

$$y=\log_b x=\frac{\ln x}{\ln b}=\frac{\ln x}{\ln b}\cdot\frac{\ln a}{\ln a}=\frac{\ln x}{\ln a}\cdot\frac{\ln a}{\ln b}=\frac{\ln x}{\ln a}\cdot\frac{1}{\frac{\ln b}{\ln a}}$$

$$=(\log_a x)\frac{1}{\log_a b}=\frac{\log_a x}{\log_a b}$$

Thus, $\log_b x=\frac{\log_a x}{\log_a b}$ (This relation is used to change logarithm from one base to another).

Example 2.12

Change $\log_4(x+1)$ to base 2

Solution

$$\log_4(x+1)=\frac{\log_2(x+1)}{\log_2 4}=\frac{\log_2(x+1)}{\log_2 2^2}=\frac{\log_2(x+1)}{2\log_2 2}=\frac{1}{2}\log_2(x+1)$$

Application activity 2.5

Find the domain of definition for each of the following functions:

1. $f(x)=\log_2\sqrt{x}$
2. $f(x)=\log_3(x^2-1)$
3. $f(x)=\log_{\frac{1}{2}}\frac{x+1}{x-4}$
4. $f(x)=\log_4\frac{x}{x^2+7x+10}$

Limit of logarithmic function with any base

Activity 2.6

Let $f(x)=\log_3 x=\frac{\ln x}{\ln 3}$ and $g(x)=\log_{\frac{1}{3}} x=\frac{\ln x}{\ln\frac{1}{3}}$

1. Evaluate limits at the boundaries of the domain of $f(x)$. Hence deduce the asymptotes, if any.
2. Evaluate limits at the boundaries of the domain of $g(x)$. Hence deduce the asymptotes, if any.

From activity 2.6 and considering $f(x)=\log_a^x$

$$\lim_{x\to 0^+} f(x)=\begin{cases}-\infty & if\ \ a>1\\ +\infty & if\ \ 0<a<1\end{cases}$$

There is a vertical asymptote $VA\equiv x=0$

$$\lim_{x\to +\infty} f(x)=\begin{cases}+\infty & if\ \ a>1\\ -\infty & if\ \ 0<a<1\end{cases}$$

There is no horizontal asymptote. In addition, no oblique asymptote.

Example 2.13

Evaluate $\lim_{x\to 0}\frac{1}{x}\log_a(1+x)$

Solution

$$\lim_{x\to 0}\frac{1}{x}\log_a(1+x)=\lim_{x\to 0}\log_a(1+x)^{\frac{1}{x}}=\log_a\left[\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right]$$

We saw that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$

Let $y=\frac{1}{x}\Rightarrow x=\frac{1}{y}$. If $x\to\infty,\ y\to 0$

$$\Rightarrow \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to +\infty}\left(1+\frac{1}{y}\right)^{y} = e$$

Then, $\log_a\left[\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right] = \log_a e$

Therefore, $\lim_{x\to 0}\frac{1}{x}\log_a(1+x) = \log_a e$

Application activity 2.6

Evaluate the following limits:

1. $\lim_{x\to 0^+}\log_2\frac{1}{x}$
2. $\lim_{x\to -2^-}\log_2\frac{x+1}{x+2}$
3. $\lim_{x\to -1^+}\log_2\frac{x+1}{x+2}$
4. $\lim_{x\to -2^+}\log_3\frac{1}{x^2-4}$

Logarithmic Differentiation

Activity 2.7

Let $f(x) = \log_2 x$

1. Find the derivative of $f(x)$
2. If $u = x^2$ is another differentiable function in x, use the rule for differentiating composite functions $(f\circ g)' = f'(g)g'$ to find the derivative of $g(x) = \log_2 u$.

From activity 2.7, as $\log_a x = \frac{\ln x}{\ln a}$, if $f(x) = \frac{\ln x}{\ln a}$

then $f'(x) = \frac{1}{x\ln a}$

Therefore, $(\log_a x)' = \frac{1}{x\ln a}$

Also, if u is another differentiable function of x, then

$$(\log_a u)' = \frac{u'}{u\ln a}$$

Example 2.14

Find the derivative of $\log_2\left(4x^2-3x\right)$

Solution

$$\left[\log_2\left(4x^2-3x\right)\right]'=\frac{\left(4x^2-3x\right)'}{\left(4x^2-3x\right)\ln 2}=\frac{8x-3}{\left(4x^2-3x\right)\ln 2}$$

Example 2.15

Find derivative of $\log_a\left(\ln\left|\sin x\right|\right)$

Solution

$$\left[\log_a\left(\ln\left|\sin x\right|\right)\right]'=\frac{\left(\ln\left|\sin x\right|\right)'}{\left(\ln\left|\sin x\right|\right)\ln a}=\frac{\left|\cot x\right|}{\left(\ln\left|\sin x\right|\right)\ln a}$$

Application activity 2.7

Differentiate each of the following functions:

1. $f(x)=\log\left(x^2+2x+1\right)$
2. $g(x)=\log_2\dfrac{x+1}{x-5}$
3. $h(x)=\log_{\frac{1}{2}}\sqrt{x^3+2x-8}$
4. $k(x)=\log_3\left(\cos\sqrt{x}\right)$

Further Logarithmic Differentiation

Activity 2.8

Let $y=\dfrac{x+1}{x-3}$

1. Introduce $\ln$ on both sides and apply the laws of logarithms.
2. Using derivative of logarithmic function, find the derivative of expression found in (1) and deduce the value of $\dfrac{dy}{dx}$.

For certain functions containing more complicated products or quotients, differentiation is often made easier if the natural logarithm of the function is taken before differentiating. This technique, called 'logarithmic differentiation' is achieved with knowledge of the:

a) laws of logarithms,
b) derivative of logarithmic functions, and
c) differentiation of implicit functions.

Example 2.16

Find derivative of $y=\frac{\sqrt[3]{x-2}}{(x+1)^2(2x-1)}$ with respect to x and hence evaluate $\frac{dy}{dx}$ for $x=3$.

Solution

From $y=\frac{\sqrt[3]{x-2}}{(x+1)^2(2x-1)}$, taking $\ln$ on both sides gives

$$\ln y=\ln\frac{\sqrt[3]{x-2}}{(x+1)^2(2x-1)}$$

Using logarithms laws, we get

$\ln y=\ln\sqrt[3]{x-2}-\ln(x+1)^2-\ln(2x-1)$ which gives

$$\ln y=\frac{1}{3}\ln(x-2)-2\ln(x+1)-\ln(2x-1)$$

Differentiating with respect to x yields $\frac{1}{y}\frac{dy}{dx}=\frac{1}{3}\frac{1}{x-2}-\frac{2}{x+1}-\frac{2}{2x-1}$

Rearranging gives, $\frac{dy}{dx}=y\left(\frac{1}{3(x-2)}-\frac{2}{x+1}-\frac{2}{2x-1}\right)$

Substituting for y gives $\frac{dy}{dx}=\frac{\sqrt[3]{x-2}}{(x+1)^2(2x-1)}\left(\frac{1}{3(x-2)}-\frac{2}{x+1}-\frac{2}{2x-1}\right)$

For $x=3$, $\frac{dy}{dx}=\frac{\sqrt[3]{3-2}}{(3+1)^2(6-1)}\left(\frac{1}{3(3-2)}-\frac{2}{3+1}-\frac{2}{6-1}\right)$

$$=\frac{1}{80}\left(\frac{1}{3}-\frac{1}{2}-\frac{2}{5}\right)=\frac{1}{80}\left(\frac{10-15-12}{30}\right)=-\frac{17}{240}$$

Application activity 2.8

Use logarithmic differentiation to find the derivative of each of the following functions:

1. $y=\frac{(x-2)(x+1)}{(x-1)(x+3)}$
2. $y=\frac{(2x-1)\sqrt{x+2}}{(x-3)\sqrt{(x+1)^3}}$
3. $y=3\theta\sin\theta\cos\theta$
4. $y=\frac{x^3\ln 2x}{e^x\sin x}$
5. $y=\frac{2x^4\tan x}{e^{2x}\tan x}$

Variations and curves of logarithmic function of any base

Activity 2.9

1. Let $f(x)=\log_2 x$
   a) From the domain of definition of $f(x)$, evaluate limits at the boundaries of the domain. Hence deduce relative asymptotes, if any.
   b) Determine the first derivative and variation of $f(x)$. Deduce the extrema, if any.
   c) Determine the second derivative and concavity of $f(x)$. Deduce the inflection points, if any.
   d) Find intersection of $f(x)$ with axes of co-ordinates.
   e) Find additional points and hence sketch the curve of $f(x)$.
2. Repeat procedures in 1) for $g(x)=\log_{\frac{1}{2}} x$

From activity 2.9, by letting $a=2$, we have $f(x)=\log_2 x$.

The curve is on figure 2.2.

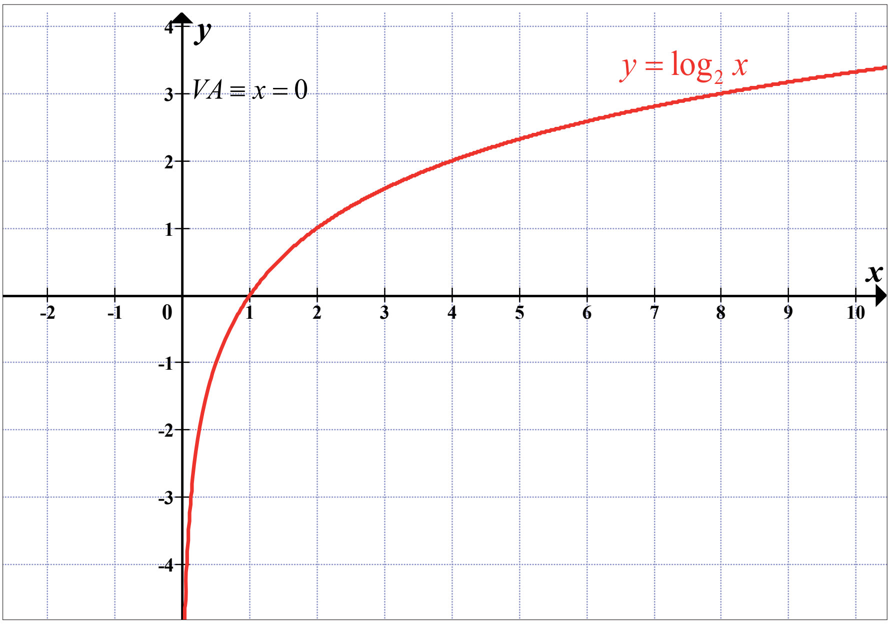


Figure 2.2. *Logarithmic function with base 2*

By letting $a = \dfrac{1}{2}$, we have $f(x) = \log_{\frac{1}{2}} x$.

The curve is given by the figure 2.3.

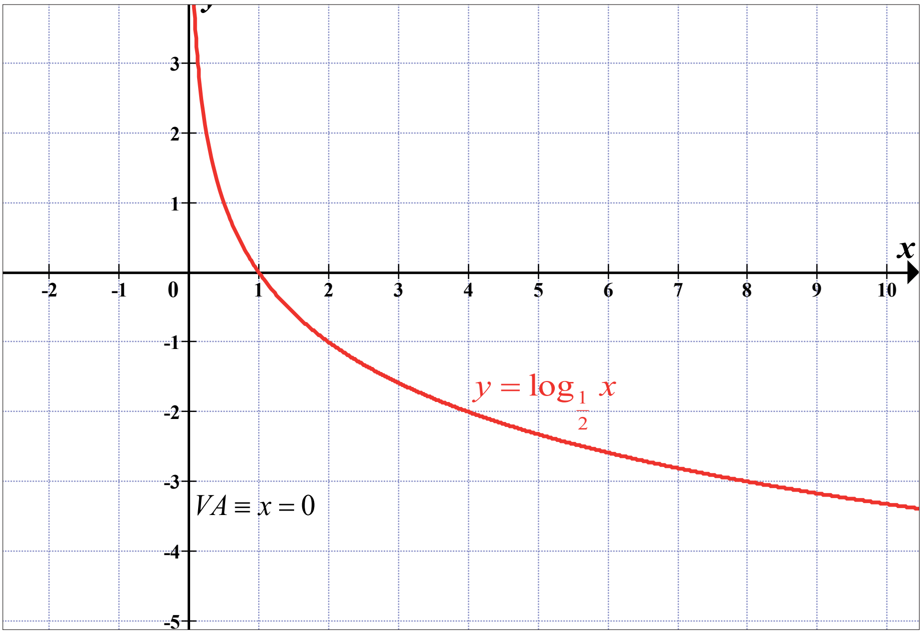


Figure 2.3. *Logarithmic function with base* $\dfrac{1}{2}$

Example 2.17

Find relative asymptotes (if any), study the variation, concavity and sketch the curve $f(x)=\log_2\sqrt{x+1}$.

Solution

Asymptotes

Domain:

$$x+1>0 \Rightarrow x>-1$$

$$Domf =]-1,+\infty[$$

$$\lim_{x\to -1} f(x)=\lim_{x\to -1}\log_2\sqrt{x+1}=-\infty$$

$x=-1$ is a vertical asymptote.

$$\lim_{x\to +\infty} f(x)=\lim_{x\to +\infty}\log_2\sqrt{x+1}=+\infty$$

No horizontal asymptote

$$\lim_{x\to +\infty}\frac{f(x)}{x}=\lim_{x\to +\infty}\frac{\log_2\sqrt{x+1}}{x}=\frac{\infty}{\infty}\quad IF$$

$$\lim_{x\to +\infty}\frac{\log_2\sqrt{x+1}}{x}=\lim_{x\to +\infty}\frac{1}{2(x+1)\ln 2}\quad [\text{Hôpital rule}]$$

$$=0$$

No oblique asymptote.

Variation

$$f'(x)=\left(\log_2\sqrt{x+1}\right)'$$

$$=\left[\frac{1}{2}\log_2(x+1)\right]'=\frac{1}{2(x+1)\ln 2}$$

Since $\forall \in Domf$, $x+1>0$ and $\ln 2>0$ then $\forall x\in Domf$, $f'(x)>0$

Hence, $\forall x\in Domf$, $f(x)$ increases.

Concavity

$$f'' = \left(\frac{1}{2(x+1)\ln 2}\right)' = -\frac{1}{2(x+1)^2 \ln 2} < 0, \forall x \in]-1,+\infty[$$

Thus, $f(x) = \log_2 \sqrt{x+1}$ is concave down $\forall x \in]-1,+\infty[$.

Variation table

x	-1	0	$+\infty$
$f'(x)$		$+$	
$f''(x)$		$-$	
$f(x)$	$-\infty$	0	$+\infty$

Curve

Intersection with axes:

Intersection with $x-axis$:

$$f(x) = 0 \Leftrightarrow \log_2 \sqrt{x+1} = 0 \text{ or } \log_2 \sqrt{x+1} = \log_2 1$$

Or $\sqrt{x+1} = 1 \Rightarrow x = 0$

Thus, $f(x) \cap ox = \{(0,0)\}$

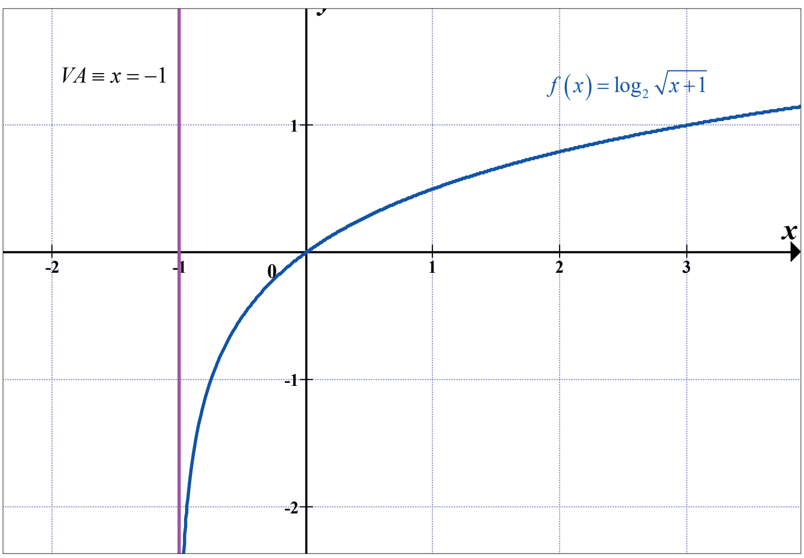
Intersection with $y-axis$:

$$f(0) = \log_2 \sqrt{0+1} = 0$$

Thus, $f(x) \cap oy = \{(0,0)\}$

Additional points

x	-0.6	-0.6	-0.4	-0.2	0	1	2	3	4	5	6
y	-1.7	-0.7	-0.4	-0.2	0	0.5	0.8	1	1.2	1.3	1.4



Application activity 2.9

For each of the following functions, find relative asymptotes (if any), study the variation concavity of the function and hence sketch the curve.

1. $f(x) = \log_2(x+1)$
2. $g(x) = \log_3(2x-4)$
3. $h(x) = \log_{\frac{1}{2}} x^2$
4. $k(x) = \log_{\frac{1}{2}} \sqrt{x}$

2.2. Exponential functions

2.2.1. Exponential function with base "e"

Domain and range of exponential functions with base "e"

Activity 2.10

Let $f(x) = \ln x$.

Suppose that $g(x)$ is the inverse function of the function. Using properties of inverse functions, find the domain and range of $g(x)$.

The function $y = \ln x$ admits an inverse function called "**Exponential function with natural base**" denoted by $y = \exp(x)$ or $y = e^x$.

From activity 2.10,

The domain of definition of $y=e^{x}$ is $]-\infty,+\infty[$ and its range is $]0,+\infty[$.

Then, $\forall x \in]0,+\infty[, y \in]-\infty,+\infty[: y=\ln x \Leftrightarrow x=e^{y}$.

Note that:

- $\forall x \in \mathbb{R}, \ln e^{x}=x$ and $\forall y \in]0,+\infty[, e^{\ln y}=y$
- $e^{0}=1$
- $e^{1}=e$

Properties

a) $e^{x}e^{y}=e^{x+y}$

b) $\left(e^{a}\right)^{n}=e^{na}$

c) $\dfrac{1}{e^{a}}=e^{-a}$

d) $\dfrac{e^{a}}{e^{b}}=e^{a-b}$

Example 2.18

Find the domain of $f(x)=e^{\sqrt{x}}$

Solution

Condition: $x \geq 0$

Thus, $Domf=[0,+\infty[$

Example 2.19

Find the domain of $g(x)=e^{\frac{x+1}{x-2}}$

Solution

Condition: $x-2 \neq 0 \Rightarrow x \neq 2$

Thus, $Domg=\mathbb{R}\setminus\{2\}$

Example 2.20

Find the domain of $h(x)=e^{\sqrt{x^{2}-1}}$

Solution

Condition: $x^2-1\geq 0 \Rightarrow x\in\left]-\infty,-1\right]\cup\left[1,+\infty\right[$

Thus, $Domh=\left]-\infty,-1\right]\cup\left[1,+\infty\right[$

Application activity 2.10

Find the domain of the following functions:

1. $f(x)=e^{\frac{2x}{x^2-7x+10}}$

2. $g(x)=e^{\frac{4x+7}{x^2-x+10}}$

3. $h(x)=\dfrac{3x+1}{1-e^{\sqrt{3x}}}$

4. $k(x)=\dfrac{e^x+1}{\log\left(e^{\sqrt{x-4}}\right)}$

Limit of exponential functions with base "e"

Activity 2.11

Let $y=e^x$

1. Complete the following tables:

x	e^x
-1	
-2	
-5	
-15	
-30	

x	e^x
1	
2	
5	
15	
30	

2. From the tables in 1), deduce $\lim\limits_{x\to-\infty} e^x$ and $\lim\limits_{x\to+\infty} e^x$. Also deduce relative asymptotes, if any.

3. Plot the graph of $y=e^x$.

From activity 2.11,

$\lim\limits_{x\to-\infty} e^x=0$ and $\lim\limits_{x\to+\infty} e^x=+\infty$

There exists horizontal asymptote: $H.A\equiv y=0$

$$\lim_{x\to-\infty}\frac{e^x}{x}=0,\ \lim_{x\to+\infty}\frac{e^x}{x}=+\infty$$

There is no oblique asymptote.

Example 2.21

Evaluate $\lim_{x\to+\infty}\frac{6e^{4x}-e^{-2x}}{8e^{4x}-e^{2x}+3e^{-x}}$

Solution

$$\lim_{x\to+\infty}\frac{6e^{4x}-e^{-2x}}{8e^{4x}-e^{2x}+3e^{-x}}=\frac{\infty-0}{\infty-\infty+0}\ IF$$

$$\lim_{x\to+\infty}\frac{6e^{4x}-e^{-2x}}{8e^{4x}-e^{2x}+3e^{-x}}=\lim_{x\to\infty}\frac{e^{4x}\left(6-e^{-6x}\right)}{e^{4x}\left(8-e^{-2x}+3e^{-5x}\right)}$$

$$=\lim_{x\to+\infty}\frac{6-e^{-6x}}{8-e^{-2x}+3e^{-5x}}=\frac{6-0}{8-0+0}=\frac{3}{4}$$

Example 2.22

Evaluate $\lim_{x\to-\infty}\frac{6e^{4x}-e^{-2x}}{8e^{4x}-e^{2x}+3e^{-x}}$

Solution

$$\lim_{x\to-\infty}\frac{6e^{4x}-e^{-2x}}{8e^{4x}-e^{2x}+3e^{-x}}=\frac{0-\infty}{0-0+\infty}\ IF$$

$$\lim_{x\to-\infty}\frac{6e^{4x}-e^{-2x}}{8e^{4x}-e^{2x}+3e^{-x}}=\lim_{x\to-\infty}\frac{e^{-x}\left(6e^{5x}-e^{-x}\right)}{e^{-x}\left(8e^{5x}-e^{3x}+3\right)}$$

$$=\lim_{x\to-\infty}\frac{6e^{5x}-e^{-x}}{8e^{5x}-e^{3x}+3}=\frac{0-\infty}{0-0+3}=-\infty$$

Application activity 2.11

Evaluate:

1. $\lim_{x\to1}e^{\frac{\sqrt[3]{x}-1}{x-1}}$
2. $\lim_{t\to-\infty}\frac{e^{6t}-4e^{-6t}}{2e^{3t}-5e^{-9t}+e^{-3t}}$
3. $\lim_{x\to+\infty}\frac{1}{2}xe^{x+1}$
4. $\lim_{x\to-\infty}\frac{1}{2}xe^{x+1}$
5. $\lim_{x\to\infty}\frac{e^{\frac{1}{x-1}}}{x}$

Derivative of exponential functions with base "e"

Activity 2.12

1. Use the method of logarithmic differentiation to find the derivative of $y = e^x$.
2. If u is another differentiable function of x, use the result obtained in 1) and the rule for differentiating composite functions to find the derivative of $y = e^u$.

Remember that this is the following:

$$f'(x) = \frac{1}{(f^{-1})'(y)}, \text{ where } y = f(x) \text{ and } (f \circ g)' = f'(g)g'$$

From activity 2.12,

$$(e^x)' = e^x$$

And if u is another differentiable function of x,

$$(e^u)' = u'e^u$$

Example 2.23

Find the derivative of the function $f(x) = \frac{1}{2}x^2 e^{x+1}$

Solution

$$f'(x) = \frac{1}{2}(2x)e^{x+1} + \frac{1}{2}x^2(1)e^{x+1} = \frac{1}{2}xe^{x+1}(x+2)$$

Example 2.24

Find the second derivative of the function $f(x) = e^{\frac{1}{x-1}}$

Solution

$$f'(x) = -\frac{1}{(x-1)^2}e^{\frac{1}{x-1}}$$

$$f''(x)=\frac{2(x-1)}{(x-1)^4}e^{\frac{1}{x-1}}-\frac{1}{(x-1)^2}\left(-\frac{1}{(x-1)^2}e^{\frac{1}{x-1}}\right)$$

$$=\frac{2x-2}{(x-1)^4}e^{\frac{1}{x-1}}+\frac{e^{\frac{1}{x-1}}}{(x-1)^4}\quad=\frac{(2x-1)e^{\frac{1}{x-1}}}{(x-1)^4}$$

Application activity 2.12

Find the derivative of:

1. $f(x)=e^{2x-1}$
2. $g(x)=e^{2x}-e^{-2x}$
3. $h(x)=e^{\tan x}$
4. $k(x)=\dfrac{e^x}{|x-1|}$

Variation and curve of exponential functions with base "e"

Activity 2.13

From the curve of $f(x)=\ln x$ (see activity 2.4), reflect it about the first bisector (the line $y=x$) to obtain new curve.

From activity 2.13, since e^x is the inverse of $\ln x$, the curve of $g(x)=e^x$ is the image of the curve of $f(x)=\ln x$ with respect to the first bisector, $y=x$.

The coordinates of the points for $f(x)=\ln x$ are reversed to obtain the coordinates of the points for $g(x)=e^x$.

The curve of $g(x)=e^x$ is as follows:

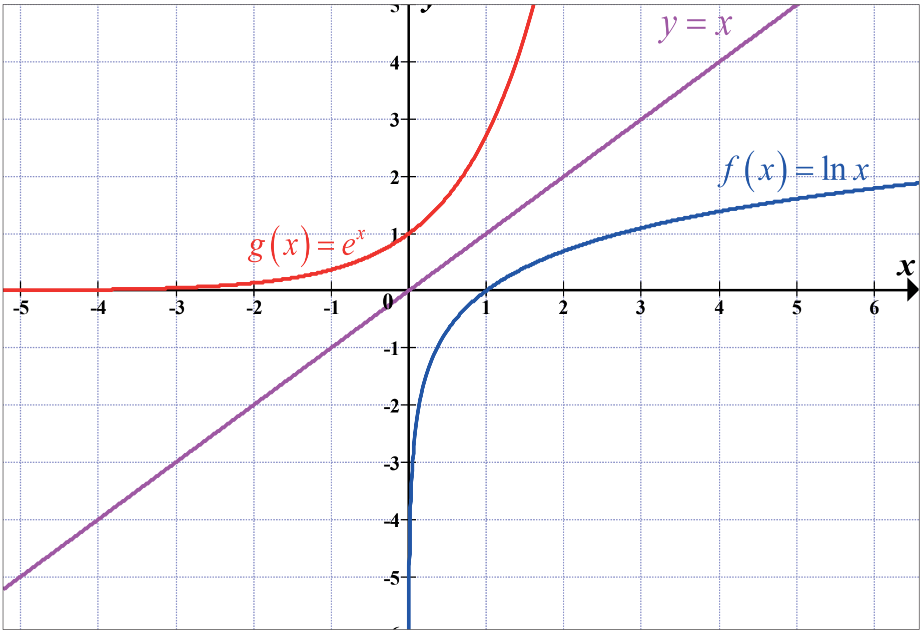


Figure 2.4. *Exponential function with base e*

Example 2.25

Given that $f(x)=\frac{1}{2}x^2e^{x+1}$. Find relative asymptotes (if any), study the variation and sketch the curve.

Solution

Asymptotes

First, we need domain of definition: $Domf=\mathbb{R}$

$\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\frac{1}{2}x^2e^{x+1}=(+\infty)(+\infty)=+\infty$, no horizontal asymptote at $x\to+\infty$

$\lim\limits_{x\to+\infty}\frac{f(x)}{x}=\lim\limits_{x\to+\infty}\frac{1}{2}xe^{x+1}=+\infty$, no oblique asymptote at $x\to+\infty$

$\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}\frac{1}{2}x^2e^{x+1}=+\infty\times 0$

Remove this indeterminate case:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{2}x^2e^{x+1} = \frac{1}{2}\lim_{x\to-\infty}\frac{x^2}{e^{-x-1}} = \frac{1}{2}\frac{-\infty}{+\infty}\ I.C.$$

$$\lim_{x\to-\infty} f(x) = \frac{1}{2}\lim_{x\to-\infty}\frac{x^2}{e^{-x-1}}$$

By Hôpital's rule

$$\lim_{x\to-\infty} f(x) = \frac{1}{2}\lim_{x\to-\infty}\frac{2x}{-e^{-x-1}} = \lim_{x\to-\infty}\frac{x}{-e^{-x-1}} = \frac{-\infty}{+\infty}\ I.C.$$

Applying again Hôpital's rule

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{1}{e^{-x-1}} = 0$$

There is horizontal asymptote $y=0$ for $x\to-\infty$. Hence, no oblique asymptote.

Also, there is no vertical asymptote according to the boundaries of the domain.

Variation

$$f'(x) = \frac{1}{2}2xe^{x+1} + \frac{1}{2}x^2e^{x+1} = \frac{1}{2}e^{x+1}\left(x^2+2x\right)$$

$f'(x) > 0 \Leftrightarrow x^2+2x>0$ or $x\in\left]-\infty,-2\right[\cup\left]0,+\infty\right[$

$f'(x) < 0 \Leftrightarrow x^2+2x<0$ or $x\in\left]-2,0\right[$

Thus, if $x\in\left]-\infty,-2\right[\cup\left]0,+\infty\right[$, $f(x)$ increases and if $x\in\left]-2,0\right[$, $f(x)$ decreases.

Curve

Intersection with axes of coordinate:

Intersection with $x-axis$;

$$f(x)=0 \Leftrightarrow \frac{1}{2}x^2e^{x+1}=0 \Rightarrow x=0$$

Thus, intersection with $x-axis$ is $\{(0,0)\}$.

Intersection with $y-axis$;

$$f(0)=\frac{1}{2}0^2e^{0+1}=0$$

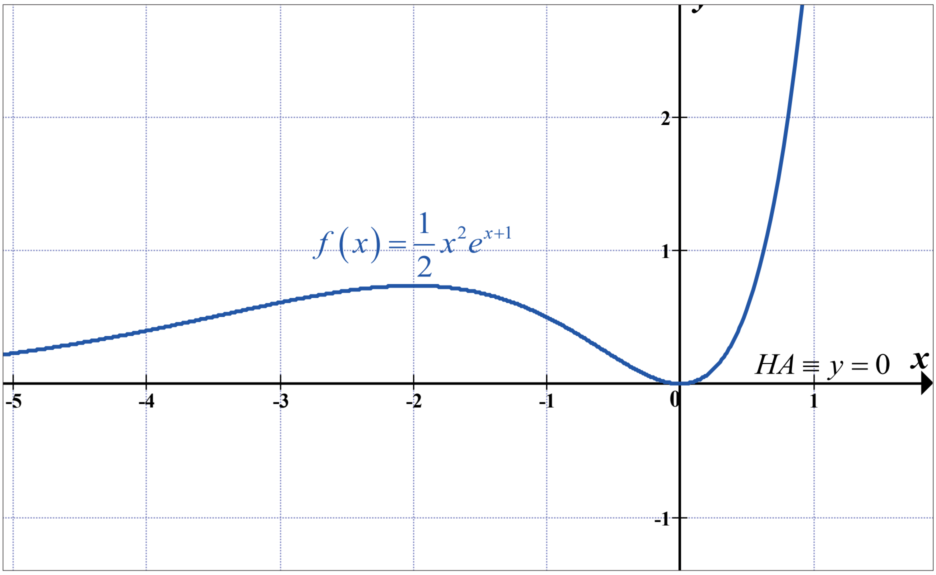
Thus, intersection with $y-axis$ is $\{(0,0)\}$.

Additional points

x	-5.00	-4.70	-4.40	-4.10	-3.80	-3.50	-3.20	-2.90	-2.60	-2.30	-2.00
$f(x)$	0.23	0.27	0.32	0.38	0.44	0.50	0.57	0.63	0.68	0.72	0.74

x	-1.70	-1.40	-1.10	-0.80	-0.50	-0.20	0.10	0.40	0.70	1.00
$f(x)$	0.72	0.66	0.55	0.39	0.21	0.04	0.02	0.32	1.34	3.69

Curve



Application activity 2.13

For each of the following functions, find relative asymptotes (if any), study the variation of the function and sketch the curve:

1. $f(x)=\frac{1}{2}e^{x+1}$
2. $g(x)=\frac{e^x}{x}$
3. $h(x)=e^{2x-3}$
4. $k(x)=\frac{e^x}{x+2}$

2.2.2. Exponential function with any base

Domain and range of exponential functions with any base

Activity 2.14

Suppose that $g(x)$ is the inverse function of the function $\log_3 x$. Using properties of inverses functions, find the domain and range of $g(x)$.

The logarithmic function with base a admit a reciprocal function called **exponential function** with base a denoted by $f(x)=a^x$.

From activity 2.14,

The domain of $f(x)=a^x$ is set of real numbers and its image is the positive real numbers.

Note that, $y=a^x \Leftrightarrow x=\log_a y$ and for $x\in\mathbb{R}$, $\log_a a^x = x$.

Properties

$\forall x, y\in\mathbb{R}, \forall a, b\in\mathbb{R}_0^+\setminus\{1\}$, we have

a) $a^x a^y = a^{x+y}$ b) $\left(a^x\right)^y = a^{xy}$ c) $(ab)^x = a^x b^x$

d) $a^{-x}=\frac{1}{a^x}$ e) $\frac{a^x}{a^y}=a^{x-y}$ f) $\left(\frac{a}{b}\right)^x=\frac{a^x}{b^x}$

Example 2.26

Find the domain of $f(x)=2^{\ln x}$

Solution

Condition: $x>0$

Thus, $Domf =]0,+\infty[$

Example 2.27

Find the domain of $f(x)=3^{\frac{x+1}{x-2}}$

Solution

Condition: $x-2 \neq 0 \Rightarrow x \neq 2$

Thus, $Domf = \mathbb{R} \setminus \{2\}$

Example 2.28

Find the domain of $f(x) = 4^{\sqrt{x^2-4}}$

Solution

Condition: $x^2 - 4 \geq 0 \Rightarrow x \in \left]-\infty, -2\right] \cup \left[2, +\infty\right[$

Thus, $Domf = \left]-\infty, -2\right] \cup \left[2, +\infty\right[$

Application activity 2.14

Find the domain of the following functions:

1. $f(x) = 3^{\frac{4}{x^2+7x+10}}$
2. $g(x) = 2^{\frac{3x+1}{x^2-x+10}}$
3. $h(x) = 4^{\sqrt{\frac{x+1}{x-3}}}$
4. $k(x) = 3^{\log\left(x^2+5x+6\right)}$

Limit of exponential functions with any base

Activity 2.15

1. Let $y = 2^x$

   a) Complete the following tables:

x	-1	-2	-5	-15	-30
2^x					

x	1	2	5	15	30
2^x					

   b) From the tables in a), deduce $\lim\limits_{x \to -\infty} 2^x$ and $\lim\limits_{x \to +\infty} 2^x$.

   Also, deduce relative asymptotes, if any.

2. Let $y=\left(\frac{1}{2}\right)^x$

a) Complete the following tables:

x	-1	-2	-5	-15	-30
$\left(\frac{1}{2}\right)^x$					

x	1	2	5	15	30
$\left(\frac{1}{2}\right)^x$					

b) From the tables in a), deduce $\lim\limits_{x\to-\infty}\left(\frac{1}{2}\right)^x$ and $\lim\limits_{x\to+\infty}\left(\frac{1}{2}\right)^x$. Also, deduce relative asymptotes, if any.

From activity 2.15,

If $a>1$, $\lim\limits_{x\to-\infty} a^x=0$ and $\lim\limits_{x\to+\infty} a^x=+\infty$

If $0<a<1$, $\lim\limits_{x\to-\infty} a^x=+\infty$ and $\lim\limits_{x\to+\infty} a^x=0$

Horizontal asymptote is $y=0$.

No vertical asymptote since the domain is the set of real numbers. In addition, there is no oblique asymptote.

Example 2.29

Evaluate $\lim\limits_{x\to-\infty} 2^{\frac{1}{x}}$

Solution

$$\lim_{x\to-\infty} 2^{\frac{1}{x}}=2^0=1$$

Example 2.30

Evaluate $\lim\limits_{x\to1} 3^{\frac{1}{x-1}}$

Solution

$$\lim_{x\to 1} 3^{\frac{1}{x-1}} = 3^{\frac{1}{0}}$$

In this case, we can easily evaluate the limits by use of the table of values below:

x	$3^{\frac{1}{x-1}}$
0	0.33
0.2	0.25
0.4	0.16
0.6	0.06
0.8	0.004
0.9	0.00001

x	$3^{\frac{1}{x-1}}$
1.1	59049
1.2	243
1.4	15
1.6	6
1.8	3.9
2	3

Now, $\lim_{x\to 1^-} 3^{\frac{1}{x-1}} = 0$ and $\lim_{x\to 1^+} 3^{\frac{1}{x-1}} = +\infty$

Hence, $\lim_{x\to 1} 3^{\frac{1}{x-1}}$ does not exist.

Indeterminate form $0^0, 1^\infty$, and ∞^0

These indeterminate forms are found in functions of the form $y = \left[f(x)\right]^{g(x)}$.

To remove these indeterminate forms, we change the function in the form $y = \left[f(x)\right]^{g(x)} = e^{g(x)\ln f(x)}$

then, $\lim_{x\to k} e^{f(x)\ln g(x)} = e^{\lim_{x\to k} f(x)\ln g(x)}$

Example 2.31

Show that $\lim_{x\to +\infty}\left(1+\frac{1}{x}\right)^x = e$

Solution

$$\lim_{x\to +\infty}\left(1+\frac{1}{x}\right)^x = 1^\infty \quad IF$$

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)} = e^{\lim\limits_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)}$$

But,

$$\lim_{x\to+\infty} x\ln\left(1+\frac{1}{x}\right) = \infty\cdot 0 \ IF$$

$$\lim_{x\to+\infty} x\ln\left(1+\frac{1}{x}\right) = \lim_{x\to+\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \frac{0}{0} \quad IF$$

$$\lim_{x\to+\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to+\infty}\frac{-\frac{1}{x^2}}{\left(1+\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)} \quad \text{[By Hôpital rule]}$$

$$= \lim_{x\to+\infty}\frac{1}{1+\frac{1}{x}} = 1$$

Thus, $\lim\limits_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e^1 = e$

Example 2.32

Evaluate $\lim\limits_{x\to 0} x^x$

Solution

$$\lim_{x\to 0} x^x = 0^0 \ IF$$

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{x\ln x} = e^{\lim\limits_{x\to 0} x\ln x} = e^0 = 1 \qquad \left[\text{since } \lim_{x\to 0} x\ln x = 0\right]$$

Alternative method

When finding limits of the function of the form $y=\left[f(x)\right]^{g(x)}$, the following relation may be used:

$$\lim_{x\to k}\left[f(x)\right]^{g(x)} = \lim_{x\to k} e^{\left[f(x)-1\right]g(x)}$$

Now lets look at example 2.31 using an alternative method.

Solution

$$\lim_{x\to 0} x^x = 0^0 \ \ IF$$

$$\lim_{x\to 0} x^x = \lim_{x\to 0} e^{(x-1)x} = e^{\lim\limits_{x\to 0}(x-1)x} = e^0 = 1 \ \ \textit{as before.}$$

Example 2.33

Evaluate $\lim\limits_{x\to +\infty}\left(\frac{x}{x+1}\right)^{x+2}$

Solution

$$\lim_{x\to +\infty}\left(\frac{x}{x+1}\right)^{x+2} = 1^{\infty}$$

$$\lim_{x\to +\infty}\left(\frac{x}{x+1}\right)^{x+2} = \lim_{x\to +\infty} e^{\left(\frac{x}{x+1}-1\right)(x+2)} = \lim_{x\to +\infty} e^{\left(\frac{x-x-1}{x+1}\right)(x+2)}$$

$$= \lim_{x\to +\infty} e^{\frac{-x-2}{x+1}} = e^{\lim\limits_{x\to +\infty}\frac{-x-2}{x+1}} = e^{-1} \qquad \left[\text{since } \lim_{x\to +\infty}\frac{-x-2}{x+1} = -1\right] = \frac{1}{e}$$

Application activity 2.15

Evaluate:

1. $\lim\limits_{x\to\infty}\left(\frac{x^2-1}{x^2+1}\right)^{\frac{x-1}{x+1}}$

2. $\lim\limits_{x\to\infty}\left(\frac{x}{x-1}\right)^{4x}$

3. $\lim\limits_{x\to 0}\left(1+x^2\right)^{\frac{1}{e^x-1-x}}$

4. $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}$

5. $\lim\limits_{x\to\infty}\left(1+\frac{k}{x}\right)^{x}$

6. $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{kx}$

Derivative of exponential functions with any base

Activity 2.16

1. Use the derivative of $\log_3 x$ and the rule of differentiating inverses functions to find the derivative of $y = 3^x$.
2. If $u = \cos x$, use the result obtained in 1) and rule for differentiation of composite functions to find the derivative of $y = 3^{\cos x}$.

Hint: $f'(x) = \dfrac{1}{\left(f^{-1}\right)'(y)}$, where $y = f(x)$ and $(f \circ g)' = f'(g)g'$

From activity 2.16,

$$\left(a^x\right)' = a^x \ln a$$

Also, if u is another differentiable function of x, we have,

$$\left(a^u\right)' = u'a^u \ln a$$

Example 2.34

Find the derivative of $f(x) = 3^{4x+2}$

Solution

$$f'(x) = (4x+2)'3^{4x+2}\ln 3 = (4\ln 3)3^{4x+2}$$

Example 2.35

Find the derivative of $f(x) = 2^{\ln x}$

Solution

$$f'(x) = (\ln x)'2^{\ln x}\ln 2 = \frac{2^{\ln x}}{x}\ln 2$$

Remarks

1. The functions; exponential of x and natural logarithm of x being reciprocal, one of another, we have $\forall y>0,\ y=e^{\ln y}$. In particular to $a^x=e^{\ln a^x}$ means $a^x=e^{x\ln a}$.

2. To study the function $y=u^v$ is the same as to study the function $y=e^{v\ln u}$ where u and v are two other functions.

3. Whenever an expression to be differentiated contains a term raised to a power which is itself a function of the variable, then logarithmic differentiation must be used. For example, the differentiation of expressions

   such as $|x|^x, (1-x)^{1-x^2}$, $\sqrt[x]{x+2}, (x)^{\sin x}$ can only be achieved using logarithmic differentiation.

Example 2.36

Find the derivative of $f(x)=x^x$.

Solution

We have $f(x)=e^{x\ln x}$

$f'(x)=\left(e^{x\ln x}\right)'=(x\ln x)'e^{x\ln x}=(1+\ln x)e^{x\ln x}=(1+\ln x)x^x$

Application activity 2.16

1. Find the derivative of the following functions:

   a) $f(x)=-2(0.3)^x$ b) $g(x)=10^x\ln x$

   c) $k(x)=\frac{1}{2}x-\frac{1}{2}x\cos 2x$ d) $k(x)=x^2(4)^{\ln x}$

2. Evaluate:

   a) $\frac{d}{dx}\left(2^{\tan x}\right)$ at $x=0$ b) $\frac{d}{dx}\left(e^{e^x}+e^{e^{e^x}}\right)$ at $x=0$

   c) $\frac{d}{dx}\left(\sqrt[x]{x-2}\right)$ at $x=3$ d) $\frac{d}{dx}\left(\sqrt[x]{x-1}\right)$ at $x=2$

Variation and curve of exponential functions with any base

Activity 2.17

1. From the curve of $f(x)=\log_2 x$, reflect it about the first bisector (the line $y=x$) to obtain a new curve.

2. From the curve of $f(x)=\log_{\frac{1}{2}} x$, reflect it about the first bisector (the line $y=x$) to obtain a new curve.

From activity 2.17, since a^x is the inverse of $\log_a x$, we can obtain a curve of a^x by symmetry with respect to the first bisector $y=x$.

Let $g(x)=2^x$, the inverse of $f(x)=\log_2 x$, the curves are as follows;

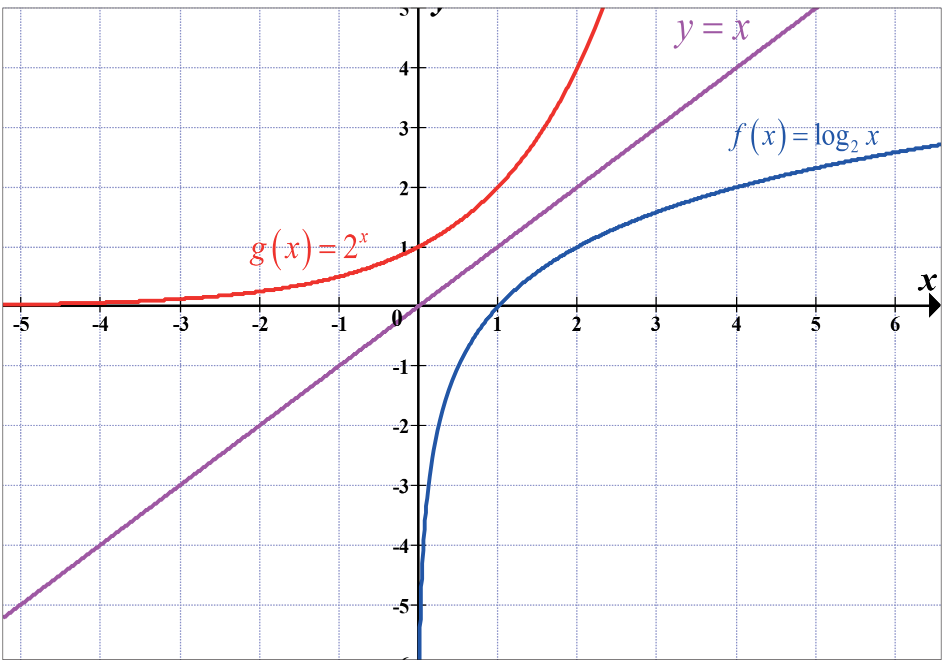


Figure 2.5. *Exponential function with base 2*

Let $g(x)=\left(\frac{1}{2}\right)^x$, the inverse of $f(x)=\log_{\frac{1}{2}} x$, the curves are as follows;

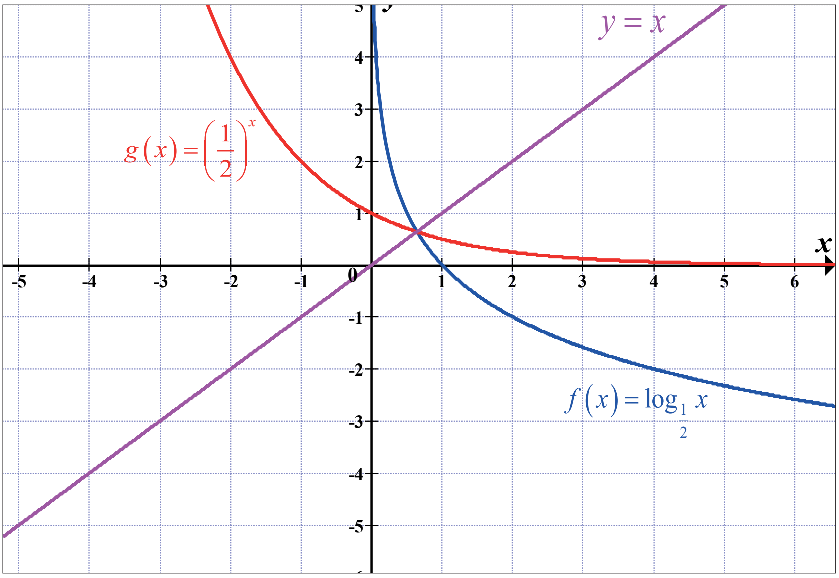


Figure 2.6. *Exponential function with base* $\frac{1}{2}$

Example 2.37

If $f(x) = 2^{5x-2}$, find the relative asymptotes (if any), study the variation and hence sketch the curve.

Solution

Asymptotes

Domain of definition: $Domf = \mathbb{R}$

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} 2^{5x-2} = 0$, there is a horizontal asymptote $y = 0$ and no oblique asymptote at $x \to -\infty$.

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} 2^{5x-2} = +\infty$$, no horizontal asymptote at $x \to +\infty$

$\lim\limits_{x\to+\infty} \frac{f(x)}{x} = \lim\limits_{x\to+\infty} \frac{2^{5x-2}}{x} = +\infty$, no oblique asymptote at $x \to +\infty$.

Also, there is no vertical asymptote according to the boundaries of the domain.

Variation

$$f'(x)=\left(2^{5x-2}\right)'=(5\ln 2)2^{5x-2}$$

$\forall x \in Domf$, $f(x)>0$ since $\ln 2>0$ and $2^{5x-2}>0$

Thus, $\forall x \in \mathbb{R}$, $f(x)$ increases.

Curve

Intersection with axes:

(i) Intersection with $x-axis$;

$f(x)=0 \Leftrightarrow 2^{5x-2}=0$ which is impossible. No intersection with $x-axis$

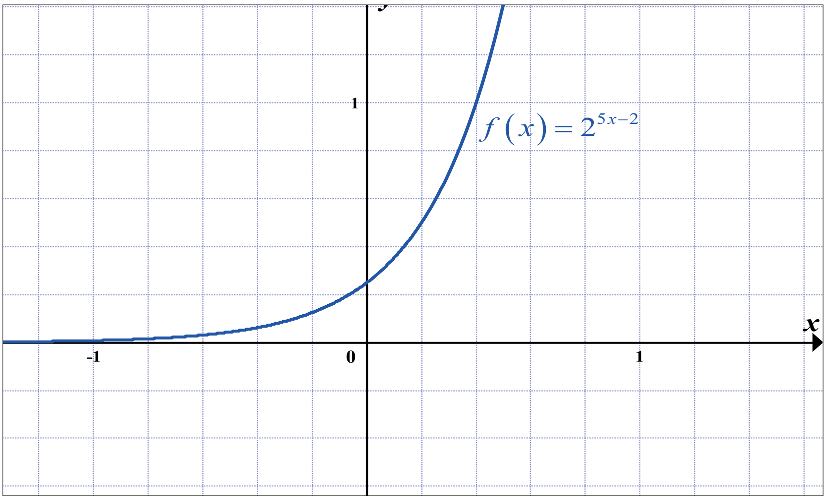
(ii) Intersection with $y-axis$;

$$f(0)=2^{-2}=\frac{1}{4}$$

Thus, intersection with $y-axis$ is $\left\{\left(0,\frac{1}{4}\right)\right\}$.

Additional points:

x	-1.2	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6
y	0.004	0.01	0.02	0.03	0.06	0.13	0.25	0.50	1.00	2.00



Application activity 2.17

For each of the following functions, find relative asymptotes (if any), study the variation of the function and sketch the curve:

1. $f(x)=2^{x+1}$
2. $g(x)=3^{-x^2+1}$
3. $h(x)=\left(\frac{1}{2}\right)^{3x}$
4. $k(x)=\left(\frac{1}{2}\right)^{x^2+2x+3}$

2.3. Applications

We have already seen in senior five, that logarithmic and exponential functions are useful where complicated calculations are involved. Now lets look at how useful they are in solving real life situations.

2.3.1. Compound interest problems

Activity 2.18

Using the library or internet if available, find out how exponential and logarithmic functions are used to solve compound interest problems. Hence solve the following problem:

If you deposit $4,000\,FRW$ into an account paying 6% annual interest compounded quarterly, how much money will be on the account after 5 years?

If P is the principal, n is the number of years, r is the interest rate per period, k is the number of periods per year, and A the total amount at the end of periods, then $A=P\left(1+\frac{r}{k}\right)^{kn}$.

Example 2.38

A 1,000 FRW deposit is made at a bank that pays 12% compounded annually. How much will be on the account at the end of 10 years?

Solution

$$A = P\left(1+\frac{r}{k}\right)^{kn}$$

$$P = 1000,\ \ r = 12\%,\ \ \ k = 1,\ n = 10$$

$$A = 1000(1+0.12)^{10} = 1000(3.1058) = 3105.8$$

Therefore, 3,105.8 FRW will be on the account at the end of 10 years.

Now if the compound interest is paid monthly for the same number of principle, we can compute the sum as follows:

Solution

In this example, the compounded is monthly, so the number of periods is $k = 12$.

$$P = 1000,\ \ r = 12\%,\ n = 12$$

$$A = 1000\left(1+\frac{0.12}{12}\right)^{12\times 10} = 1000(3.3004) = 3300.4$$

Therefore, 3,300.4 FRW will be on the account at the end of 10 years.

Application activity 2.18

1. If you deposit 6,500 FRW into an account paying 8% annual interest compounded monthly, how much money will be on the account after 7 years?
2. How much money would you need to deposit today at 9% annual interest compounded monthly to have 12,000 FRW on the account after 6 years?
3. If you deposited 5,000 FRW into an account paying 6% annual interest compounded monthly, how long will you wait until there is 8,000 FRW on the account?
4. If you deposited 8,000 FRW into an account paying 7% annual interest compounded quarterly, how long will you wait until there is 12,400 FRW on the account?
5. At 3% annual interest compounded monthly, how long will it take to double the amount of money deposited in question 4?

2.3.2. Mortgage amount problems

Activity 2.19

Using the library or internet if available, find out how exponential and logarithmic functions are used to solve mortgage amount problems. Hence, solve the following problem:

Suppose you wanted to take out a mortgage for 100,000 FRW with monthly payments at 9%, but you could only afford 800 FRW per month payments. How long would you have to make payments to pay off the mortgage, and how much interest would you pay for this payment period?

There is a relationship between the mortgage amount, the number of payments, the amount of the payment, how often the payment is made, and the interest rate. The following formula illustrates the relationship: $$P=\frac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}.$$

Where

$P=$ the payment , $r=$ the annual rate , $M=$ the mortgage amount ,

$t=$ the number of years and $n=$ the number of payments per year .

The payment P required to pay off a loan of M Francs borrowed for n payment periods at a rate of interest i per payment period is

$$P=M\left[\frac{i}{1-(1+i)^{-n}}\right] \text{ where } i=\frac{r}{n}$$

Example 2.39

a) What is the monthly payment on a mortgage of 75,000 FRW with an 8% interest rate that runs for

(i) 20 years (ii) 25 years?

b) How much interest is paid in each case?

Solution

a) (i) **20 years**

$$P=\frac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}$$

$$M=75000,\ \ r=8\%,\ t=20,\ \ n=12$$

We are solving for P (the monthly payment for the 20 years)

$$P=\frac{\frac{0.08\times 75000}{12}}{1-\left(1+\frac{0.08}{12}\right)^{-12\times 20}}=627.33$$

The monthly payment will be 627.33 FRW

After 20 years of payments (20×12 months), you will have paid $20\times 12\times 627.33=150,559.20$

(ii) **25 Years**

$$P=\frac{\frac{rM}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}$$

$$M=75000,\ \ r=8\%,\ t=25,\ \ n=12$$

We are solving for P (the monthly payment for the 25 years)

$$P=\frac{\frac{0.08\times 75000}{12}}{1-\left(1+\frac{0.08}{12}\right)^{-12\times 25}}=578.86$$

The monthly payment will be 578.86 FRW.

After 25 years of payments (25×12 months), you will have paid $25\times 12\times 578.86=173,658$.

b) (i) Everything over the initial 75,000 FRW is interest.

Therefore, after 20 years, you will have paid

$150,559.20\,FRW - 75,000\,FRW = 75,559.20\,FRW$ in interest.

(ii) Everything over the initial 75000 FRW is interest.
Therefore, after 25 years, you will have paid

$173,658\,FRW - 75,000\,FRW = 98,658\,FRW$ in interest.

Application activity 2.19

1. A person borrowed 1,200,000 FRW for the purchase of a car. If his monthly payment is 31,000 FRW on a 5-year mortgage, find the total amount of interest.
2. If a house is sold for 3,000,000 FRW and the bank requires 20% down payment, find the amount of the mortgage.
3. Mr Peter bought a car. After paying the down payment, the amount of the loan is 400,000 FRW with an interest rate of 9% compounded monthly. The term of the loan is 3 years. How much is the monthly payment?
4. Suppose you need to take out a mortgage of 100,000 FRW. All you can afford for monthly payments is 800 FRW. You will retire in 25 years; therefore, the longest you can make these payments is 25 years. What interest rate would you need to take out a mortgage of 100,000 FRW and pay it back in 300 monthly payments of 800 FRW.

2.3.3. Population growth problems

Activity 2.20

Using the library or internet if available, find out how exponential and logarithmic functions are used to solve population growth problems. Hence, solve the following problem:

Betty is investigating the growth in the population of a certain type of bacteria in her flask. At the start of day 1, there are 1,000 bacteria in flask. The population of bacteria grows exponentially at the rate of 50% per day. Find the population of bacteria in her flask at the start of day 5.

If P_0 is the population at the beginning of a certain period and $r\%$ is the constant rate of growth per period, the population for n periods will be $P_n = P_0\left(1+r\right)^n$.

Example 2.40

The town of Grayrock had a population of 10,000 in 1960 and 12,000 in 1970.

a) Assuming an exponential growth model, estimate the population in 1980.

b) What is the doubling time for the town's population?

Solution

a) For exponential growth model $P_n = P_0(1+r)^n$.

Let in 1960, have $P_0 = 10,000$.

Thus, in 1970, we have $P_{10} = 12,000$ while in 1980, we have P_{20}.

$$P_{10} = 12,000 \Leftrightarrow 12,000 = P_0(1+r)^{10}$$

$$\Leftrightarrow 12,000 = 10,000(1+r)^{10}$$

$$\Leftrightarrow 12 = (1+r)^{10}$$

$$\Leftrightarrow \sqrt[10]{12} = 1+r$$

$$\Rightarrow r = 0.018399376$$

$$P_n = P_0(1+r)^n \Rightarrow P_{20} = 10,000(1.018399376)^{20} = 14,400$$

The population in 1980 is 14,000.

b) The doubling time for the town's population means the time for which $P_n = 2P_0$,

$$P_n = 2P_0 \Leftrightarrow 2P_0 = P_0(1+r)^n \Leftrightarrow (1+r)^n = 2$$

$$\Rightarrow n = \frac{\ln 2}{\ln(1+r)} \Rightarrow n = 38\ years$$

Hence, the doubling time for the town's population is 38 years.

Application activity 2.20

1. The population, P, of an island t years after January 1st 2016 is given by this formula $P = 4200 \times (1.04)^t$
   a) What was the population of the island on January 1st 2016?
   b) What is the constant rate?
   c) Work out the population of the island on January 1st 2021.
2. The population of a city increased by 5.2% for the year 2014. At the beginning of 2015 the population of the city was 1,560,000. Betty assumes that the population will continue to increase at a constant rate of 5.2% each year. Use Betty's assumption to;
   a) Estimate the population of the city at the beginning of 2017. Give your answer corrected to 3 significant figures.
   b) Work out the year in which the population of the city will reach 2,000,000.

2.3.4. Depreciation value problems

Activity 2.21

Using the library or internet if available, find out how exponential and logarithmic functions are used to solve depreciation value problems. Hence, solve the following problem:

During an experiment, a scientist notices that the number of bacteria halves every second. If there were 2.3×10^{30} bacteria at the start of the experiment, how many bacteria were left after 5 seconds. Give your answer in standard form corrected to two significant figures.

Depreciation (or **decay**) is negative growth. If V_0 is the value at a certain time, and $r\%$ is the rate of depreciation per period, the value V_t at the end of t periods is $V_t = V_0(1-r)^t$.

Example 2.41

If you start a biology experiment with 5,000,000 cells and 45% of the cells are dying every minute, how long will it take to have less than 1,000 cells?

Solution

Using the equation, $V_t = V_0\left(1-r\right)^t$;

$V_t = 1,000,\ V_o = 5,000,000,\ r = 0.45$

$\Rightarrow 1,000 = 5,000,000\left(1-0.45\right)^t$

$\Leftrightarrow \dfrac{1,000}{5,000,000} = \left(1-0.45\right)^t$

$\Rightarrow 0.0002 = \left(0.55\right)^t$

$\Rightarrow \ln 0.0002 = \ln\left(0.55\right)^t \qquad \left[\text{taking } \ln \ \textit{both} \text{ sides}\right]$

$\Rightarrow \ln 0.0002 = t\ln 0.55$

$\Rightarrow t = \dfrac{\ln 0.0002}{\ln 0.55} \Rightarrow t \approx 14.2$

It will take about 14.2 minutes for the cell population to drop below a 1,000 count.

Application activity 2.21

1. In a certain experiment, the number of bacteria reduces by a quarter each second. If the number of bacteria initially was X, write a formula that can be used to calculate the number of bacteria, V, remaining after t seconds.

2. The population of a particular town on July 1, 2011 was 20,000. If the population decreases at an average annual rate of 1.4%, how long will it take for the population to reach 15,300?

2.3.5. Earthquake problems

Activity 2.22

Using the library or internet if available, find out how exponential and logarithmic functions are used to solve earthquake problems. Hence, solve the following problem:

How many times stronger is an earthquake with a magnitude of 8 than an earthquake with a magnitude of 6?

In 1935, Charles Richter defined the magnitude of an earthquake to be

$M = \log \frac{I}{S}$ where I is the intensity of the earthquake (measured by the amplitude of a seismograph reading taken $100\,km$ from the epicentre of the earthquake) and S is the intensity of a "standard earthquake" (whose amplitude is $1\,micron = 10^{-4}\,cm$).

The magnitude of a standard earthquake is $M = \log \frac{S}{S} = \log 1 = 0$.

Richter studied many earthquakes that occurred between 1900 and 1950. The largest had magnitude of 8.9 on the Richter scale, and the smallest had magnitude 0. This corresponds to a ratio of intensities of 800,000,000, so the Richter scale provides more manageable numbers to work with.

Example 2.42

Early in the century, the earthquake in San Francisco registered 8.3 on the Richter scale. In the same year, another earthquake was recorded in South America that was four times stronger. What was the magnitude of the earthquake in South America?

Solution

$$M_{San\ Francisco} = \log \frac{I_{San\ Francisco}}{S} = 8.3$$

$$\log \frac{I_{San\ Francisco}}{S} = 8.3$$

$$M_{South\ America} = \log \frac{I_{South\ America}}{S}$$

$$I_{South\ America} = 4I_{San\ Francisco}$$

$$M_{South\ America} = \log \frac{4I_{San\ Fransisco}}{S}$$

Solve for $M_{South\ America}$,

$$M_{South\ America} = \log \frac{4I_{San\ Francisco}}{S}$$

$$= \log 4I_{San\ Francisco} - \log S$$

$$= \log 4 + \log I_{San\ Francisco} - \log S = \log 4 + \log \frac{I_{San\ Francisco}}{S}$$

$$= \log 4 + 8.3 = 8.90205999133$$

$$M_{South\ America} = 8.9$$

The intensity of the earthquake in South America was 8.9 on the Richter scale.

Example 2.43

A recent earthquake in San Francisco measured 7.1 on the Richter scale. How many times more intense was the San Francisco earthquake described in Example 2.42?

Solution

The intensity of each earthquake was different. Let I_1 represent the intensity of the early earthquake and I_2 represent the latest earthquake.

$$First : 8.3 = \log \frac{I_1}{S}$$

$$Second : 7.1 = \log \frac{I_2}{S}$$

What you are looking for is the ratio of the intensities: $\frac{I_1}{I_2}$.

$$\log \frac{I_1}{S} - \log \frac{I_2}{S} = 8.3 - 7.1$$

$$\Rightarrow \log I_1 - \log S - (\log I_2 - \log S) = 1.2$$

$$\Rightarrow \log I_1 - \log S - \log I_2 + \log S = 1.2$$

$$\Rightarrow \log I_1 - \log I_2 = 1.2 \Rightarrow \log \frac{I_1}{I_2} = 1.2$$

$$\Rightarrow \log \frac{I_1}{I_2} = \log 10^{1.2} \Rightarrow \frac{I_1}{I_2} = 10^{1.2} \Rightarrow \frac{I_1}{I_2} \approx 16 \Rightarrow I_2 \approx 16 I_1$$

The early earthquake was 16 times as intense as the later earthquake.

Application activity 2.22

1. An earthquake monitoring station measured the amplitude of the waves during a recent tremor as being 100,000 times as large as A_0, the smallest detectable wave. How high did this earthquake measure on the Richter scale?
2. An earthquake is measured with a wave amplitude 392 times as great as A_0. What is the magnitude of this earthquake using the Richter scale, to the nearest tenth?
3. The San Francisco earthquake of 1989 measured 6.9 on the Richter scale. The Alaska earthquake of 1964 measured 8.5.
   a) How many times as intense as the San Francisco earthquake was the Alaska earthquake?
   b) Calculate the magnitude of an earthquake that is twice as intense as the 1989 San Francisco earthquake.
4. How much intense is an earthquake measuring 6.5 on the Richter scale than one measuring 6.4?

2.3.6. Carbon-14 dating problems

Activity 2.23

Using library or internet if available, find out how exponential and logarithmic functions are used to work out the age of organic material. Hence solve the following problem:

If you had a fossil that had 10 percent carbon-14 compared to a living sample. How old is that fossil?

Carbon dating is used to work out the age of organic material – in effect, any living thing. The technique hinges on carbon-14, a radioactive isotope of the element that, unlike other more stable forms of carbon, decays at a steady rate. Organisms capture a certain amount of carbon-14 from the atmosphere when they are alive. By measuring the ratio of the radio isotope to non-radioactive carbon, the amount of carbon-14 decay can be worked out, thereby giving an age for the specimen in question.

The half-life of a substance is the time it takes for half the original amount of that substance to decay. It is only a property of substances that decay at a rate proportional to their mass. Through research, scientists have agreed that the half-life of Carbon-14 is approximately 5,700 years.

A formula to calculate how old a sample is by carbon-14 dating is:

$$t = \frac{\ln\left(\frac{N_f}{N_0}\right)}{-0.693} \times t_{\frac{1}{2}}$$

where $\frac{N_f}{N_0}$ is the percent age (fraction) of carbon-14 in the sample compared to the amount in living tissue, and $t_{\frac{1}{2}}$ is the half-life of carbon-14 which is $5,730 \pm 30$ years.

Example 2.44

A scrap of paper taken from the Dead Sea Scrolls was found to have a $\frac{^{14}C}{^{12}C}$ ratio of 0.795 times that found in plants living

today. Estimate the age of the scroll.

Solution

$$t = \frac{\ln(0.795)}{-0.693} \times 5,700 = 1,887 \; years \; old$$

Example 2.45

A chemist determines that a sample of petrified wood has a carbon-14 decay rate of 6.00 counts per minute per gram. What is the age of the piece of wood in years? (The decay rate of carbon-14 in fresh wood today is 13.60 counts per minute per gram, and the half life of carbon-14 is 5,730 years).

Solution

$$t = \frac{\ln\left(\frac{6.00}{13.6}\right)}{-0.693} \times 5,730 = 6,766 \; years \; old$$

Example 2.46

Using dendrochronology (a technique that uses tree rings to determine age), tree materials dating back 10,000 years have been identified. Assuming you had a sample of such a tree in which the number of Carbon-14 decay events was 15.3 decays per minute before decomposition, what would the decays per minute be in the present day?

Solution

$$\text{Using; } t = \frac{\ln\left(\frac{N_f}{N_0}\right)}{-0.693} \times t_{\frac{1}{2}}$$

$$\Rightarrow 10,000 = \frac{\ln\left(\frac{N_f}{N_0}\right)}{-0.693} \times 5,730 \Rightarrow \ln\left(\frac{N_f}{N_0}\right) = \frac{10,000 \times (-0.693)}{5730} = -1.21$$

$$\ln\left(\frac{N_f}{N_0}\right) = (-1.21)\ln e; \quad \text{since } \ln e = 1$$

$$\Rightarrow \ln\left(\frac{N_f}{N_0}\right) = \ln e^{-1.21} \Rightarrow \frac{N_f}{N_0} = e^{-1.21} \Rightarrow N_0 = \frac{N_f}{e^{-1.21}}$$

But $N_f = 15.3$

Then, $N_0 = \frac{15.3}{e^{-1.21}} = 4.6$

$$N_o = \frac{15.3}{e^{-1.21}} = 51.3$$

Application activity 2.23

1. The carbon-14 decay rate of a sample obtained from a young tree is 0.296 disintegration per second per gram of the sample. Another wood sample prepared from an object recovered at an archaeological excavation gives a decay rate of 0.109 disintegration per second per gram of the sample. What is the age of the object?
2. The Carbon-14 content of an ancient piece of wood was found to have three tenths of that in living trees (indicating 70% of the Carbon-14 had decayed). How old is the piece of wood?
3. Carbon-14 is used to determine the age of ancient objects. If a sample today contains 0.060 mg of carbon-14, how much carbon-14 must have been present in the sample 11,430 years ago?

4. A living plant contains approximately the same isotopic abundance of Carbon-14 as does atmospheric carbon dioxide. The observed rate of decay of Carbon-14 from a living plant is 15.3 disintegrations per minute per gram of carbon. How much disintegration per minute per gram of carbon will be measured from a 12900-year-old sample? (The half-life of Carbon-14 is 5730 years.)
5. All current plants have a Carbon-14 count of 15.3 cpm. How old is a wooden artifact if it has a count of 9.58 cpm?
6. You read that a fossil dinosaur skull has been found in Montana and that it has been carbon-14 dated to be 73 million years old. Provide two (2) scientifically-based reasons to explain why Carbon-14 dating cannot do this.

The other applications of logarithms are in many scientific contexts. Some of which include:

a) Measure of Sound

Sound is measured in a logarithmic scale using a unit called a **decibel**.

The formula looks similar to the Richter scale; $d = 10\log\left(\frac{P}{P_0}\right)$ where P is the power or intensity

of the sound and P_0 is the weakest sound that the human ear can hear.

Example 2.47

One hot water pump has a noise rating of 50 decibels. One dishwasher, however, has a noise rating of 62 decibels. Determine how many times the dish washer more intense than the hot water pump noise.

Solution

$$50 = 10\log\left(\frac{h}{P_0}\right) \Leftrightarrow 10^5 = \frac{h}{P_0} \Rightarrow h = 10^5 P_0$$

$$\Leftrightarrow 62 = 10\log\left(\frac{d}{P_0}\right) \Leftrightarrow 6.2 = \log\left(\frac{d}{P_0}\right) \Leftrightarrow 10^{6.2} = \frac{d}{P_0} \Rightarrow d = 10^{6.2} P_0$$

Then,

$$\frac{d}{h} = \frac{10^{6.2} P_0}{10^5 P_0} = 10^{1.2}$$

Thus, the dishwasher's noise is $10^{1.2}$ (or about 15.85) times as intense as the hot water pump.

b) Measure of acidity

The measure of acidity of a liquid is called the **pH** of the liquid. This is based on the amount of hydrogen ions, H^+ in the liquid.

The formula for pH is $pH = -\log\left[H^+\right]$ where $\left[H^+\right]$ is the concentration of hydrogen ions, given in a unit called mol/L ("moles per litre"; Recall that one mole is 6.022×10^{23} molecules or atoms).

Liquids with a low pH (below 7) are more acidic than those with a high pH. Water, which is neutral (neither acidic nor alkaline, the opposite of acidic) has a pH of 7.0.

Example 2.48

If lime juice has a pH of 1.7, what is the concentration of hydrogen ions (in mol/l^{-1}) in lime juice, to the nearest hundredth?

Solution

$$pH = -\log\left[H^+\right]$$

$$1.7 = -\log x \Leftrightarrow -1.7 = \log x \Leftrightarrow x = 10^{-1.7} \Leftrightarrow x = 0.02$$

The concentration of hydrogen ions in lime juice is 0.02.

Unit Summary

1. Logarithmic function

- **Domain of definition and range**

The Natural logarithm of x is denoted as $\ln x$ or $\log_e x$ and defined on positive real numbers, $]0,+\infty[$, its range is all real numbers.

$\forall x \in]1,+\infty[$, $\ln x > 0$ and $\forall x \in]0,1[$, $\ln x < 0$

The equation $\ln x = 1$ has, in interval $]0,+\infty[$,

a unique solution, a rational number $2.71828182845904523536 0....$

This number is denoted by e.

Hence, $\ln x = 1 \Leftrightarrow x = e$.

Generally $e = \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x$

- **Limits on boundaries**

Logarithmic function $f(x) = \ln x$ being defined on $]0,+\infty[$,
$\lim_{x\to+\infty} \ln x = +\infty$ and $\lim_{x\to 0^+} \ln x = -\infty$

From, $\lim_{x\to+\infty} \ln x = +\infty$, we deduce that there is no

horizontal asymptote.

From $\lim_{x\to 0^+} \ln x = -\infty$, we deduce that there exists a

vertical asymptote with equation $VA \equiv x = 0$.

- **Derivative of natural logarithmic functions or logarithmic derivative**

$x \in \mathbb{R}_0^+, (\ln x)' = \frac{1}{x}$ and $(\ln x)' > 0$

Also, if u is differentiable function at x then,

$$(\ln u)' = \frac{u'}{u}$$

With certain functions containing more complicated products and quotients, differentiation is often made easier if the logarithm of the function is taken before differentiating.

- **Domain and limits on boundaries of a logarithmic function with any base**

Logarithm function of a real number x with base a is
a function f denoted $f(x) = \log_a x$ and defined by

$$\log_a x = \frac{\ln x}{\ln a}, \; x \in \mathbb{R}_0^+, \; a \in \mathbb{R}_0^+ \setminus \{1\} \; .$$

$$\forall x \in \mathbb{R}_0^+, \log_a x = y \Leftrightarrow x = a^y$$

$$\lim_{x\to 0^+} f(x) = \begin{cases} -\infty & \text{if } a > 1 \\ +\infty & \text{if } 0 < a < 1 \end{cases}$$

There is a vertical asymptote $VA \equiv x = 0$

$$\lim_{x \to +\infty} f(x) = \begin{cases} +\infty & if \ \ a > 1 \\ -\infty & if \ \ 0 < a < 1 \end{cases}$$

There is no horizontal asymptote nor oblique asymptote.

- **Logarithmic Differentiation**

If $f(x) = \log_a x$, then $f'(x) = \dfrac{1}{x \ln a}$

Also, if u is another differentiable function of x,

then,

$$(\log_a u)' = \frac{u'}{u \ln a}$$

2. Exponential functions

- **Exponential function with base "e"**

Domain and range of exponential functions with base "e";

The domain of definition of $y = e^x$ is $-\infty + \infty|$ and its range is $]0, +\infty[$.

Then, $\forall x \in]0, +\infty[, y \in]-\infty, +\infty[: y = \ln x \Leftrightarrow x = e^y$.

- **Limit of exponential functions with base "e"**

$\lim\limits_{x \to -\infty} e^x = 0$ and $\lim\limits_{x \to +\infty} e^x = +\infty$

There exists horizontal asymptote: $H.A \equiv y = 0$

- **Derivative of exponential functions with base "e"**

$\forall x \in \mathbb{R}, \ (e^x)' = e^x$

If u is another differentiable function at x,

$(e^u)' = u'e^u$

Remarks

a) $\forall y > 0,\ y = e^{\ln y}$

In particular, $a^x = e^{\ln a^x}$ means $a^x = e^{x\ln a}$.

Hence, to study the function $y = u^v$ is the same as to study the function $y = e^{v\ln u}$ where u and v are two other functions.

b) Whenever an expression to be differentiated contains a term raised to a power which is itself a function of the variable, then, logarithmic differentiation must be used. For example, the differentiation of expressions such as $|x|^x, (1-x)^{1-x^2}$, $\sqrt[x]{x+2}, (x)^{\sin x}$ and so on can only be achieved using logarithmic differentiation.

c) Population growth

If P_0 is the population at the beginning of a certain period and $r\%$ is the constant rate of growth per period, the population after n periods will be $P_n = P_0(1+r)^n$.

d) Depreciation value

Depreciation (or decay) is negative growth. If V_0 is the value at a certain time, and $r\%$ is the rate of depreciation per period, the value V_t at the end of t periods is $V_t = V_0(1-r)^t$.

e) Earthquake

Charles Richter defined the magnitude of an earthquake to be

$M = \log\frac{I}{S}$ where I is the

intensity of the earthquake (measured by the amplitude of a seismograph reading taken $100\,km$ from the epicentre of the earthquake) and S is the intensity of a "standard earthquake" (whose amplitude is $1\,micron = 10^{-4}\,cm$).

f) Carbon-14 dating

Carbon dating is used to work out the age of organic material – in effect, any living thing. By measuring the ratio of the radio isotope to non-radioactive carbon, the amount of carbon-14 decay can be worked out, thereby giving an age for the specimen in question.

Through research, scientists have agreed that the half-life of Carbon-14 is approximately 5700 years.

A formula to calculate how old a sample is by carbon-14 dating is:

$$t = \frac{\ln\left(\frac{N_f}{N_0}\right)}{-0.693} \times t_{\frac{1}{2}}$$

where $\frac{N_f}{N_0}$ is the percent of carbon-14 in the sample compared to the amount in living tissue, and $t_{\frac{1}{2}}$ is the half-life of carbon-14 which is $5,730 \pm 30$ years.

End of unit assessment

In questions 1-8, find the domain of definition.

1. $f(x) = \ln(x^2 - 1) - 4\ln(4x - 1) + 2\ln 2$
2. $f(x) = 2\log_2 x + \log_x 2 - 3$
3. $f(x) = \ln(x-2)(x-1) - \ln(2x+8)$
4. $f(x) = \ln\left|x^2 - 4x + 1\right|$
5. $f(x) = \log_x 5 - \log_5 x$
6. $f(x) = 4^{\frac{1}{x}} \cdot 16^{\frac{1}{x+2}} - 64^{\frac{1}{x+1}}$
7. $f(x) = \frac{1}{2}x^2 e^{x+1}$
8. $f(x) = e^{\frac{1}{x-1}}$

In questions 9-22, evaluate the given limits.

9. $\lim\limits_{x \to 1^-} e^{\frac{1}{x-1}}$
10. $\lim\limits_{x \to 1^+} \frac{e^x}{x-1}$
11. $\lim\limits_{x \to -\infty} \frac{e^x}{1-x}$
12. $\lim\limits_{x \to +\infty} \ln(e^x - 1)$

13. $\lim\limits_{x\to+\infty} \frac{\ln\left(e^x-1\right)}{x}$

14. $\lim\limits_{x\to+\infty} \ln\left(e^x-1\right)-x$

15. $\lim\limits_{x\to-\infty} \log\left(x^2-9\right)$

16. $\lim\limits_{x\to-\infty} 3^{x+2}$

17. $\lim\limits_{x\to-\infty} \left(\frac{1}{3}\right)^{x+2}$

18. $\lim\limits_{x\to+\infty} \frac{x^7+x^5+x^3}{\left(\frac{1}{2}\right)^x}$

19. $\lim\limits_{x\to0^-} \frac{2}{3+4^{\frac{1}{x}}}$

20. $\lim\limits_{x\to+\infty} \left(1+\frac{1}{x+2}\right)^{x-1}$

21. $\lim\limits_{x\to+\infty} \left(1-\frac{2}{3x}\right)^x$

22. $\lim\limits_{n\to+\infty} \left(\frac{2n^2}{3n+1}\right)^{\frac{-3n^2+2}{5n-3}}$

In questions 23-30, find the derivative of each function.

23. $f(x)=\left(x^2-2x+2\right)e^x$

24. $f(x)=\ln\left(\tan\frac{x}{2}\right),\ 0\le x\le 2\pi$

25. $f(x)=\ln(\ln x)$

26. $f(x)=\ln\left(x+\sqrt{x^2+a^2}\right),\ a\in\mathbb{R}$

27. $f(x)=\frac{e^x-1}{e^x+1}$

28. $f(x)=\ln\frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}+x}$

29. $f(x)=\left(\frac{x}{a}\right)^{\sqrt{x}},\ a\in\mathbb{R}_0^+$

30. $f(x)=(\cos x)^x$

In questions 31-34, find relative asymptotes (if any) , study the variation of the function and sketch the curve.

31. $f(x)=xe^{x+}$

32. $f(x)=2^{x^2-}$

33. $f(x)=\frac{e^x}{|x-1}$

34. $f(x)=\frac{4-\ln x}{x^2}$

35. Suppose that you are observing the behavior of cell duplication in a laboratory. If in one of the experiments, you started with 1,000,000 cells and the cell population decreased by ten percent every minute.

 a) Write an equation with base (0.9) to determine the number of cells after t minutes.

 b) Determine how long it would take the population to reach a size of 10 cells.

36. A city in Texas had a population of 75,000 in 1970 and a population of 200,000 in 1995. The growth between the years 1970 and 1995 followed an exponential pattern of the form $f(t) = A \times e^{\alpha t}$.

 a) Find the values of A and α.

 b) Using the given model, estimate the population for the year 2010.

37. An \$1000 deposit is made at a bank that pays 12% compounded weekly. How much will you have on your account at the end of 10 years?

38. What is the monthly payment on a mortgage of \$75000 with an 8% interest rate that runs for 30 years? How much interest is paid over 30 years?

39. Suppose a bank offers you a 10% interest rate on a 20-year mortgage to be paid back with monthly payments. Suppose the most you can afford to pay in monthly payments is \$700. How much of a mortgage could you afford?

40. Suppose that you are observing the behavior of cell duplication in a laboratory. If in one of the experiments, you started with one cell and the cell population is tripling every minute.

 a) Write an equation with base 3 to determine the number of cells after one hour.

 b) Determine the number of cells after one hour.

41. Suppose that you are observing the behavior of cell duplication in a lab. In one experiment, you started with 100,000 cells and observed that the cell population decreased by half every minute.

 a) Write an equation (model) with base $\frac{1}{2}$

 to determine the number of cells (size of population) after t minutes.

 b) Determine the number of cells after 10 minutes.

42. In 1946, an earthquake struck Vancouver Island. It had an amplitude that was $10^{7.3}$ times A_o.

    a) What was the earthquake's magnitude on the Richter scale?

    b) The strongest earthquake in Canada struck Haida Gwaii, off the BC coast, in 1949. It had a Richter reading of 8.1. How many times as great as A_o was its amplitude?

43. The 2011 Tohoku earthquake, which occurred off the coast of Japan, measured 9.03 on the Richter scale. Calculate the magnitude of an earthquake that is one-quarter as intense as this earthquake. Round off to the nearest hundredth.

44. A common ingredient in cola drinks is phosphoric acid, the same ingredient found in many rust removers.

    a) If a cola drink has a pH of 2.5, what is the hydrogen ion concentration of the cola drink?

    b) Milk has a pH of 6.6. How many times more acidic than milk is a cola drink? Round off to the nearest whole number.

45. Refer to the decibel scale in the figure below, how many times as intense as the sound of normal conversation is the sound of a rock concert?

0 dB	Threshold for human hearing
10 dB	
20 dB	Whisper
30 dB	Quiet library
40 dB	Quiet conversation
50 dB	
60 dB	Normal conversation
70 dB	Hair dryer
80 dB	
90 dB	Lawnmower
100 dB	
110 dB	Car horn
120 dB	Rocket concert

46. Sounds that are utmost 95,000 times as intense as a whisper are considered safe, no matter how long or how often you hear them. The sound level of a whisper is about 20 dB. What is the maximum sound level that is considered safe? Round off to the nearest decibel.

Unit 3

Taylor and Maclaurin's Expansions

Introductory activity

Suppose that we need to complete the table below.

Angle, x	0^0	1^0	2^0	3^0	4^0	5^0
$\sin x$						

For $x = 0^0$ is very easy since this angle is a remarkable angle. But, what about other angles; $1^0, 2^0, 3^0, 4^0, 5^0$? How can we find their sine without using sine button on scientific calculator?

A series is a summation of the terms of a sequence. Finite series is a summation of a finite number of terms and an infinite series has an infinite number of terms and an upper limit of infinity.

Taylor series is a representation of a function as an infinite sum of terms that are calculated from the values at a single point. The concept of a Taylor series was formally introduced by the ENGLISH Mathematician BROOK TAYLOR in 1715. The special case of Taylor series is Maclaurin series.

Objectives

By the end of this unit, a student will be able to:

- Find the sum of a given series.
- Find the Taylor series of a given function.
- Find the Maclaurin series of a given function.
- Use Maclaurin series to;
  - » calculate limits,
  - » approximate the values of some constants,
  - » approximate an irrational number,
  - » approximate logarithmic number,
  - » approximate trigonometric number of an angle,
  - » approximate the roots of a given equation.

Maclaurin's series is used when finding limits of some functions, approximate irrational number like $\sqrt{2}$, finding trigonometric number of an angle, …

3.1. Generalities on series

3.1.1. Finite series

Activity 3.1

Suppose we want to find the sum, $\sum_{k=1}^{n} u_k$, of a series $u_1 + u_2 + \ldots + u_n$ where the terms follow a certain pattern.

1. If $u_k = f(k) - f(k+1)$, where $f(k)$ is some function of k.

   For $k = 1$, $u_1 = f(1) - f(2)$

   For $k = 2$, $u_2 = f(2) - f(3)$

   Continue in this way up to $k = 5$. Find the general relation for $k = n-1$ and $k = n$.

2. Add the terms obtained in 1) to obtain the sum of the series.

The sum of a number of terms where the terms follow a definite pattern is called **series**. If the terms are finite, then the series is said to be **finite** and if they are infinite the series is said to be **infinite**.

A finite series is an expression of the form $u_1 + u_2 + u_3 + \ldots + u_n$ or in sigma notation we write $\sum_{k=1}^{n} u_k$, where the index of summation, k, takes consecutive integer values from the lower limit, 1, to the upper limit, n. The terms $u_1, u_2, u_3, \ldots, u_n$ are called terms of the series and the term u_n is the general term.

We saw, in senior five, how to find the sum of n terms of an arithmetic progression and sum of n terms of a geometric progression. Arithmetic and geometric series are standard series. But now the question is how can we find the sum of a series which is not a familiar standard series? In this case, the method of difference is usually used.

From activity 3.1, we can write:

$$\sum_{k=1}^{n} u_n = f(1) - f(n+1)$$

Example 3.1

Find the sum of the series $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\ldots+\frac{1}{n(n+1)}$

Solution

Clearly, this is not a familiar standard series, such as an arithmetic or geometric series. Therefore, we apply the method of differences to obtain the required sum.

Now, the k^{th} term, u_k, is given by $\frac{1}{k(k+1)}$. We now need to try to split up u_k.

The only sensible way to do this is to express $\frac{1}{k(k+1)}$ in partial fractions.

Let $\frac{1}{k(k+1)}=\frac{A}{k}+\frac{B}{k+1}$.

The constants A and B are used because k and $k+1$ are linear factors.

Then,

$$\frac{1}{k(k+1)}=\frac{A(k+1)+Bk}{k(k+1)}$$

$$\Leftrightarrow 1=Ak+A+Bk$$

$$\Leftrightarrow 1=k(A+B)+A$$

Comparing the coefficients of k, $A+B=0$ and comparing the constants, $A=1$.

Hence, $B=-1$ and $\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}$

From which we see that;

$f(k)=\frac{1}{k}$ and $f(k+1)=\frac{1}{k+1}$

Now, writing down the series term by term we have;

$k=1 \qquad \frac{1}{2} \qquad =\frac{1}{1}-\frac{1}{2}$

$k=2 \qquad \frac{1}{6} \qquad =\frac{1}{2}-\frac{1}{3}$

$k=3 \qquad \frac{1}{12} \qquad =\frac{1}{3}-\frac{1}{4}$

$\vdots$

$k=n-1 \qquad \frac{1}{(n-1)n}=\frac{1}{n-1}-\frac{1}{(n-1)+1}$

$k=n \qquad \frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$

Adding, some terms will be canceled and we remain with $\sum_{k=1}^{n}\frac{1}{k(k+1)}=1-\frac{1}{n+1}=\frac{n}{n+1}$

Thus, $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$

Example 3.2

Find the sum of the series $\frac{1}{5}+\frac{1}{21}+\frac{1}{45}+\ldots+\frac{1}{(2n-1)(2n+3)}$

Solution

Let $u_k=\frac{1}{(2k-1)(2k+3)}$

Now expressing $\frac{1}{(2k-1)(2k+3)}$ in terms of partial fractions,

we have; $\frac{1}{(2k-1)(2k+3)}=\frac{A}{2k-1}+\frac{B}{2k+3}$

$$\frac{1}{(2k-1)(2k+3)}=\frac{A(2k+3)+B(2k-1)}{(2k-1)(2k+3)} \Leftrightarrow 1=A(2k+3)+B(2k-1)$$

Comparing the coefficients of k, $2A+2B=0$, so $A=-B$. Comparing the constants terms, $1=3A-B$. Hence, $A=\frac{1}{4}$ and

$B=-\frac{1}{4}$.

Thus, $\frac{1}{(2k-1)(2k+3)}=\frac{1}{4(2k-1)}-\frac{1}{4(2k+3)}$

Now substituting for $r=1,2,3,\ldots$; we obtain

$$k=1 \qquad \frac{1}{5} \qquad =\frac{1}{4}-\frac{1}{20}$$

$$k=2 \qquad \frac{1}{21} \qquad =\frac{1}{12}-\frac{1}{28}$$

$$k=3 \qquad \frac{1}{45} \qquad =\frac{1}{20}-\frac{1}{36}$$

$$k=4 \qquad \frac{1}{77} \qquad =\frac{1}{28}-\frac{1}{44}$$

$$\vdots$$

$$k=n-2 \qquad \frac{1}{(2n-5)(2n-1)}=\frac{1}{4(2n-5)}-\frac{1}{4(2n-1)}$$

$$k=n-1 \qquad \frac{1}{(2n-3)(2n+1)}=\frac{1}{4(2n-3)}-\frac{1}{4(2n+1)}$$

$$k=n \qquad \frac{1}{(2n-1)(2n+3)}=\frac{1}{4(2n-1)}-\frac{1}{4(2n+3)}$$

Adding, some terms will be canceled:

$$\frac{1}{5}+\frac{1}{21}+\frac{1}{45}+\ldots+\frac{1}{(2n-2)(2n+3)}=\frac{1}{4}+\frac{1}{12}-\frac{1}{4(2n+1)}-\frac{1}{4(2n+3)}$$

$$=\frac{1}{4}\left(1+\frac{1}{3}-\frac{1}{2n+1}-\frac{1}{2n+3}\right)=\frac{1}{4}\left[1+\frac{1}{3}-\left(\frac{1}{2n+1}+\frac{1}{2n+3}\right)\right]$$

$$=\frac{1}{4}\left[1+\frac{1}{3}-\left(\frac{(2n+3)+(2n+1)}{(2n+1)(2n+3)}\right)\right]=\frac{1}{4}\left[\frac{4}{3}-\left(\frac{4(n+1)}{(2n+1)(2n+3)}\right)\right]$$

$$=\frac{1}{3}-\left(\frac{n+1}{(2n+1)(2n+3)}\right)$$

Thus, $\frac{1}{5}+\frac{1}{21}+\frac{1}{45}+...+\frac{1}{(2n-2)(2n+3)}=\frac{1}{3}-\left(\frac{n+1}{(2n+1)(2n+3)}\right)$

Application activity 3.24

Find the sums of the following series:

1. $\sum_{r=1}^{n}\frac{1}{r(r+1)}$
2. $\sum_{r=1}^{n}\frac{1}{4r^2-1}$
3. $\sum_{r=1}^{n}r(r+1)(r+2)$
4. $\sum_{r=1}^{n}\frac{1}{r+1}$

3.1.2. Infinite series

Activity 3.2

Consider the series $S_n=\frac{7}{10}+\frac{7}{10^2}+\frac{7}{10^3}+\frac{7}{10^4}+...+\frac{7}{10^n}+...$

1. Multiply both sides of the given series by $\frac{1}{10}$.
2. Subtract the series obtained in 1) from the given series to find the expression of S_n in terms of n.
3. Evaluate limit of the S_n obtained in 2) as $n\to+\infty$.

The purpose here is to discuss sums $u_1+u_2+u_3+...+u_n+...$ that contain infinitely many terms. The most familiar examples of such sums occur in the decimal representation of real numbers.

For example, when we write $\frac{1}{3}$ in the decimal form we have;

$$\frac{1}{3}=0.3333...$$

$$\Rightarrow\frac{1}{3}=0.3+0.03+0.003+0.0003+...$$

$$=\frac{3}{10}+\frac{3}{10^2}+\frac{3}{10^3}+\frac{3}{10^4}+...$$

Since it is impossible to add up infinitely many numbers, we will deal with infinite sums by means of a limiting process involving sequences.

An infinite series is an expression of the form $u_1 + u_2 + u_3 + \ldots + u_n + \ldots$ or in sigma notation $\sum_{n=1}^{+\infty} u_n$. The terms $u_1, u_2, u_3, \ldots$ are called terms of the series.

To carry out this summation process, we proceed as follows:

Let s_n denote the sum of the first n terms of the series. Thus,

$$s_1 = u_1$$

$$s_2 = u_1 + u_2$$

$$s_3 = u_1 + u_2 + u_3$$

$$\vdots$$

$$s_n = u_1 + u_2 + u_3 + \ldots + u_n = \sum_{k=1}^{n} u_k$$

The number s_n is called the n^{th} partial sum of the series and the sequence $\{s_n\}_{n=1}^{+\infty}$ is called the sequence of partial sums.

Example 3.3

What are the partial sums of the series $\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \ldots$

Solution

The partial sums are

$$s_1 = \frac{3}{10}$$

$$s_2 = \frac{3}{10} + \frac{3}{10^2}$$

$$s_3 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3}$$

$$s_4 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4}$$

$$\vdots$$

As n increases, the partial sum $s_n = u_1 + u_2 + u_3 + ... + u_n$ includes more and more terms of the series. Thus, if s_n tends towards a limit as $n \to +\infty$, it is reasonable to view this limit as the sum of all the terms in the series. This suggests the following definition:

Let $\{s_n\}$ be the sequence of partial sums of the series $\sum_{k=1}^{+\infty} u_k$.

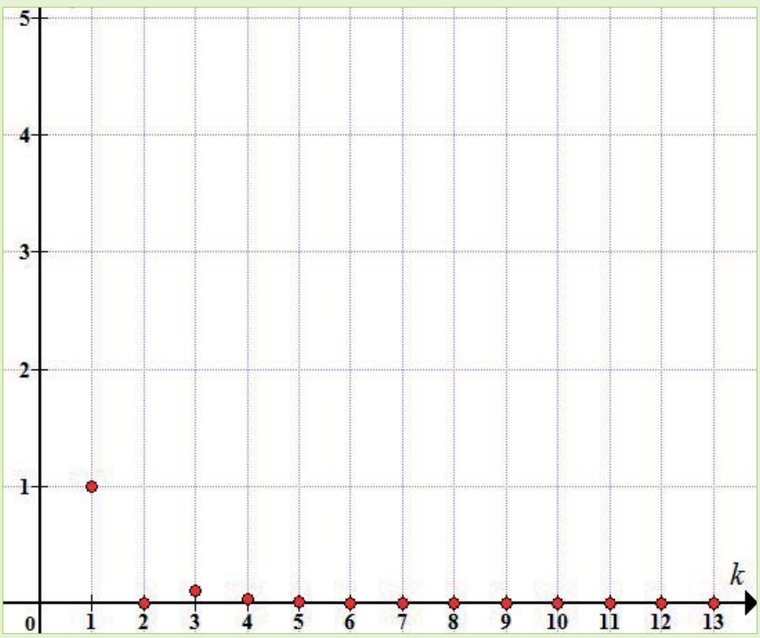
If the sequence $\{s_n\}$ converges to a limit S, then the series is said to **converge** and S is called the **sum** of the series. We denote this by writing $S = \sum_{k=1}^{+\infty} u_k$.

If the sequence of partial sums of a series diverges, then the series is said to **diverge**. A divergent series has no sum.

Geometric interpretation of a convergence series and a divergence series: graphical approach.

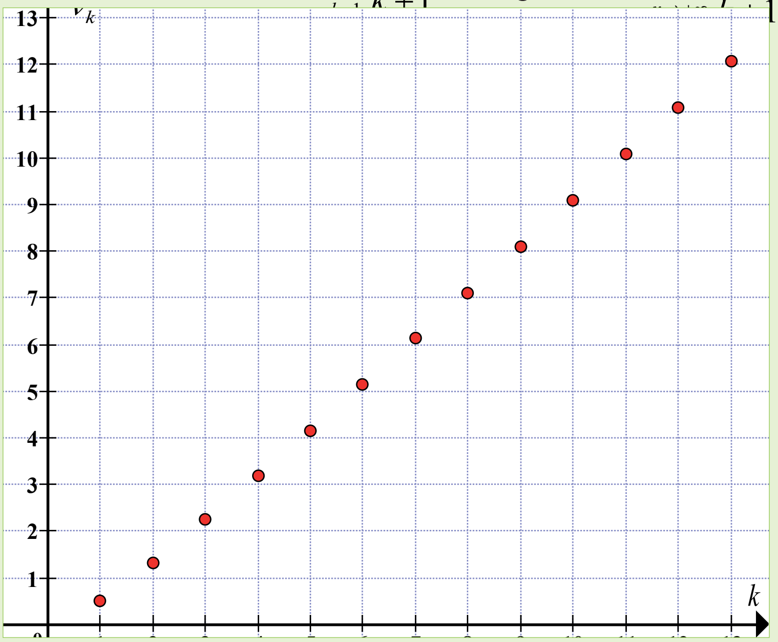
Consider two series $u_k = \sum_{k=1}^{\infty} \frac{1}{3^{k-1}}$ and $v_k = \sum_{k=1}^{\infty} \frac{k^2}{k+1}$.

- The series $u_k = \sum_{k=1}^{\infty} \frac{1}{3^{k-1}}$ converges to 0 since $\lim_{k \to +\infty} \frac{1}{3^{k-1}} = 0$,



The graph of u_k shows that as k increases, u_k is 0.

- The series $v_k = \sum_{k=1}^{\infty} \frac{k^2}{k+1}$ diverges since $\lim_{k \to +\infty} \frac{k^2}{k+1} = +\infty$



The graph shows that as k increases, v_k also increases.

Example 3.4

Find the sum of the series $\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \ldots + \frac{3}{10^k} + \ldots$

Solution

Here, the n[th] partial sum is $s_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \ldots + \frac{3}{10^n}$ $\quad (1)$

The problem of calculating the limit is complicated by the fact that the number of terms in (1) changes with n.

First, we multiply both sides of (1) by $\frac{1}{10}$ to obtain

$$\frac{1}{10} s_n = \frac{3}{10^2} + \frac{3}{10^3} + \frac{3}{10^4} + \frac{3}{10^5} + \ldots + \frac{3}{10^{n+1}} \qquad (2)$$

And then subtracting (2) from (1) we obtain;

$$s_n - \frac{1}{10} s_n = \frac{3}{10} - \frac{3}{10^{n+1}}$$

or $\frac{9}{10}s_n = \frac{3}{10}\left(1-\frac{1}{10^n}\right)$ or $s_n = \frac{1}{3}\left(1-\frac{1}{10^n}\right)$

Now, taking the limit we have;

$$S = \lim_{n\to+\infty} s_n = \lim_{n\to+\infty} \frac{1}{3}\left(1-\frac{1}{10^n}\right) = \frac{1}{3}$$

Thus, $\frac{3}{10}+\frac{3}{10^2}+\frac{3}{10^3}+\frac{3}{10^4}+\ldots+\frac{3}{10^k}+\ldots = \frac{1}{3}$

Notice

The series in above example is a geometric series with initial term $u_1 = \frac{3}{10}$ and common ratio $r = \frac{1}{10}$. We can also find the sum $s_n = \frac{3}{10}+\frac{3}{10^2}+\frac{3}{10^3}+\frac{3}{10^4}+\ldots+\frac{3}{10^n}$ using the method we saw, in senior five, on the sum of n terms of a geometric sequence and then find the limit there after.

That is,

$$s_n = \frac{u_1\left(1-r^n\right)}{1-r}$$

$$= \frac{\frac{3}{10}\left[1-\left(\frac{1}{10}\right)^n\right]}{1-\frac{1}{10}} = \frac{\frac{3}{10}\left[1-\frac{1}{10^n}\right]}{\frac{9}{10}} = \frac{3}{10}\left(1-\frac{1}{10^n}\right)\times\frac{10}{9}$$

$$= \frac{1}{3}\left(1-\frac{1}{10^n}\right)$$

And then,

$$S = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \frac{1}{3}\left(1-\frac{1}{10^n}\right) = \frac{1}{3}$$

Example 3.5

Determine whether the series $1-1+1-1+1-1+\ldots$ converges or diverges. If it converges, find the sum.

Solution

The partial sums are $s_1 = 1, s_2 = 1-1 = 0, s_3 = 1-1+1 = 1, s_4 = 1-1+1-1 = 0$ and so forth.

Thus, the sequence of partial sums is $1, 0, 1, 0, \ldots$.

Since this sequence is divergent, the given series diverges and consequently it has no sum.

Example 3.6

Determine whether the series $\sum_{k=1}^{+\infty} \frac{1}{k(k+1)}$ converges or diverges. If it converges, find the sum.

Solution

First, we find the nth partial sum of the series which is

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots + \frac{1}{n(n+1)}$$

From example 3.1, we have seen that $\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$

So $S = \lim_{n \to +\infty} \left(1 - \frac{1}{n+1}\right) = 1$

And therefore, $\sum_{k=1}^{+\infty} \frac{1}{k(k+1)} = 1$

Notice

Recall that, in senior five, a geometric series $u_1 + u_1 r + u_1 r^2 + \ldots + u_1 r^{k-1} + \ldots$, $u_1 \neq 0$ converges if $|r| < 1$ and diverges if $|r| \geq 1$.

In case of convergence, the sum is $\frac{u_1}{1-r}$.

That is, $u_1 + u_1 r + u_1 r^2 + \ldots + u_1 r^{k-1} + \ldots = \frac{u_1}{1-r}$ for $|r| < 1$.

Example 3.7

The series $5+\frac{5}{4}+\frac{5}{4^2}+...+\frac{5}{4^{k-1}}+...$ is a geometric series with $u_1=5$ and $r=\frac{1}{4}$. Find the sum of the series.

Solution

Since $|r|=\left|\frac{1}{4}\right|=\frac{1}{4}<1$, the series converges and the sum is

$$\frac{u_1}{1-r}=\frac{5}{1-\frac{1}{4}}=\frac{20}{3}$$

Example 3.8

Find the rational number represented by the repeating decimal $0.784784784...$

Solution

Here, we can write $0.784784784...=0.784+0.000784+0.000000784+...$

So the given decimal is the sum of geometric series with $u_1=0.784$ and $r=0.001$.

Thus,

$$0.784784784...=\frac{u_1}{1-r}=\frac{0.784}{1-0.001}=\frac{0.784}{0.999}=\frac{784}{999} \qquad \left[since\ |r|<1\right]$$

Application activity 3.25

1. Find the sums of the following series:

a) $\sum_{r=1}^{n} r$ b) $\sum_{r=1}^{n} r^2$

c) $\sum_{r=1}^{n} r^3$ d) $\sum_{r=1}^{n} r(r+1)$

2. Find the rational number represented by each of the following repeating decimal:

a) $0.27272727...$ b) $0.8333333...$

c) $0.1237373737...$

Tests for convergence of series

Activity 3.3

1. Evaluate:

a) $\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}$ for $u_n=\frac{3^n+1}{5^n}$ b) $\lim\limits_{n\to\infty}\sqrt[n]{u_n}$ for $u_n=\frac{n}{3^n}$

2. Consider the series $\sum_{n=1}^{\infty}\frac{1}{2n-1}$. Find a series $\sum_{n=1}^{\infty}b_n$ such that $\frac{1}{2n-1}\le b_n$.

Comparison test

Let $\sum_{n=1}^{\infty}a_n$ be a series with positive terms;

a) $\sum_{n=1}^{\infty}a_n$ converges if there exists a convergent series $\sum_{n=1}^{\infty}b_n$ such that $a_n\le b_n$ for all $n>N$, where N is some positive integer.

b) $\sum_{n=1}^{\infty}a_n$ diverges if there exists a divergent series $\sum_{n=1}^{\infty}c_n$ such that $a_n\ge c_n$ for all $n>N$, where N is some positive integer.

Limit comparison test

If the series $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ are two series with positive terms, and $\lim\limits_{n\to\infty}\frac{a_n}{b_n}$ is finite, both series converge or both diverge.

The ratio test

Let $\sum_{n=1}^{\infty} u_n$ be a series with positive terms and let $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L$,

then,

a) the series converges if $L < 1$

b) the series diverges if $L > 1$

c) the series may or may not converge if $L = 1$ (i.e. the test is inconclusive).

The n^{th} root test

Let $\sum_{n=1}^{\infty} u_n$ be a series with positive terms and let $\lim_{n\to\infty} \sqrt[n]{u_n} = L$,

then,

a) the series converges if $L < 1$

b) the series diverges if $L > 1$

c) the test is inconclusive $L = 1$.

Example 3.9

Use the comparison test to show that the series $\sum_{n=1}^{\infty} \frac{2n+1}{n}\left(\frac{1}{2}\right)^n$ is convergent.

Solution

$\frac{2n+1}{n}\left(\frac{1}{2}\right)^n \le 3\left(\frac{1}{2}\right)^n$ and $\sum_{n=1}^{\infty} 3\left(\frac{1}{2}\right)^n$ is a convergent geometric series converging to 3.

Therefore, the series $\sum_{n=1}^{\infty} \frac{2n+1}{n}\left(\frac{1}{2}\right)^n$ converges by comparison.

Example 3.10

Use the ratio test to determine whether or not the series $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ is convergent.

Solution

$$\frac{u_{n+1}}{u_n}=\frac{\dfrac{((n+1)!)^2}{(2(n+1))!}}{\dfrac{(n!)^2}{(2n)!}}=\frac{((n+1)!)^2}{(2(n+1))!}\times\frac{(2n)!}{(n!)^2}$$

$$=\frac{(n+1)(n+2)}{(2n+2)(2n+1)}=\frac{n+1}{2(2n+1)}$$

$\lim\limits_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\frac{n+1}{2(2n+1)}=\frac{1}{4}<1$ and so the series converges.

Example 3.11

Use the n^{th} root test to determine whether or not the series $\sum\limits_{n=1}^{\infty}\frac{3^n}{n^2 2^n}$ is convergent.

Solution

$\lim\limits_{n\to\infty}\sqrt[n]{\frac{3^n}{n^2 2^n}}=\lim\limits_{n\to\infty}\frac{3}{2\left(\sqrt[n]{n}\right)^2}=\frac{3}{2}>1$ and so the series does not converge.

Application activity 3.26

Use either the ratio or the n^{th} root test to determine which of the following series converges or diverges.

1. $\sum\limits_{n=1}^{\infty}\frac{n^4}{3^n}$
2. $\sum\limits_{n=1}^{\infty}\frac{n^5}{5^n}$
3. $\sum\limits_{n=1}^{\infty}\frac{2^{3n}}{3^{2n}}$
4. $\sum\limits_{n=1}^{\infty}\frac{n!}{2^n}$
5. $\sum\limits_{n=1}^{\infty}\frac{n!}{(2n+1)!}$
6. $\sum\limits_{n=1}^{\infty}\frac{(n+1)(n+2)}{n!}$

3.2. Power series

Activity 3.4

1. Use n^{th} root test to determine the condition for x for which the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ is convergent.
2. Use ratio test to determine whether the series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ converges or not.

Power series is like an infinite polynomial. It has the form

$$\sum_{n=0}^{\infty}a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots + a_n(x-c)^n + \ldots$$

Here, c is any real number and a series of this form is called a power series centred at c.

Let $f(x)=\sum_{n=0}^{\infty}a_n(x-c)^n$ be the function defined by this power series. $f(x)$ is only defined if the power series converges, so we will consider the domain of the function f to be the set of x values for which the series converges. There are three possible cases:

- The power series converges at $x=c$. Here the radius of convergence is zero.
- The power series converges of all x, i.e $]-\infty,+\infty[$. Here the radius of convergence is infinity.
- There is a number R called the **radius of convergence** such that the series converges for all $c-R<x<c+R$ and the series diverges outside this interval.

Example 3.12

Find the radius of convergence of $\sum_{n=0}^{\infty}\frac{x^n}{n!}$

Solution

First, note that this is a power series centred at $c=0$, and the coefficient $a_n = \dfrac{1}{n!}$.

We will use ratio test to find the radius of convergence:

$$\lim_{n\to\infty}\left|\frac{\dfrac{x^{n+1}}{(n+1)!}}{\dfrac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0$$

Since the ratio test implies that the series converges and final answer, i.e. 0 does not depend on x, we see that the series will converge for all x and thus the radius of convergence is infinite.

Example 3.13

Find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n (x-2)^n}{3^n}$

Solution

This is a power series centred at $c=2$, and the coefficient $a_n = \dfrac{(-1)^n}{3^n}$.

$$\lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}(x-2)^{n+1}}{3^{n+1}}}{\dfrac{(-1)^n (x-2)^n}{3^n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)(x-2)}{3}\right| = \left|\frac{x-2}{3}\right|$$

The series will converge if $\left|\dfrac{x-2}{3}\right| < 1$ or if $|x-2| < 3$.

From which the radius of convergence $R=3$ and the series converge to $\forall x$ such that $-1 < x < 5$ since

$$c - R < x < c + R$$

$$\Leftrightarrow 2-3 < x < 2+3$$

$$\Leftrightarrow -1 < x < 5$$

Application activity 3.27

For each of the following power series, determine the values of x for which the series converges and the radius of convergence.

1. $\sum_{n=0}^{\infty}(x+2)^n$
2. $\sum_{n=0}^{\infty}n(3x+1)^n$
3. $\sum_{n=0}^{\infty}(-1)^n(2x+3)^n$
4. $\sum_{n=0}^{\infty}\frac{(x-1)^n}{2^n}$
5. $\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$
6. $\sum_{n=0}^{\infty}\frac{x^n}{\sqrt{n^2+1}}$
7. $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$
8. $\sum_{n=0}^{\infty}\frac{(n+1)(x+5)^n}{3^n}$
9. $\sum_{n=0}^{\infty}n!(x-3)^n$
10. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(n+1)x^n}{n!}$

3.2.1. Taylor and Maclaurin series

Activity 3.5

Suppose that $f(x)$ is any function that can be represented by a power series: $f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n$ or

$$f(x)=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+c_4(x-a)^4+...+c_n(x-a)^n+... \quad (1)$$

1. Find $f(a)$
2. Find $f'(a)$
3. Find $f''(a)$. Deduce, using factorial notation, the value of c_2
4. Find $f'''(a)$. Deduce, using factorial notation, the value of c_3.
5. Find $f^{(iv)}(a)$. Deduce, using factorial notation, the value of c_4
6. Find $f^{(iv)}(a)$. Deduce, using factorial notation, the value of c_4
7. From results obtained in 1) to 5), deduce the value of $f^{(n)}(a)$. Deduce the value of c_n.

8. Substitute the values of $c_0, c_1, c_2, c_3, c_4, ..., c_n$, obtained in 1) to 6), in relation (1) to obtain new relation.

From activity 3.5, if $f(x)$ is a function defined on the open interval $]a,b[$, and which can be differentiated $(n+1)$ times on $]a,b[$, then the equality

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)+...+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+R_{n+1}(x),$$

$x\in\mathbb{R}$ and $x_0\in\,]a,b[$

is called **Taylor's formula**; where $n!$ denotes the factorial of n, $f^{(n)}(x_0)$ denotes the n^{th} derivative of $f(x)$ evaluated at point x_0 and $R_{n+1}(x)$ denotes **remainder function**.

The polynomial

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)+...+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

is called **the n^{th} degree Taylor polynomial of f at x_0**.

If $\lim\limits_{n\to\infty} R_{n+1}(x)=0$ for some terms in x, then the infinite series

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

is called the **Taylor series** for $f(x)$.

Notice

For special case $x_0=0$, the Taylor series becomes

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$

$$=f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+...+\frac{f^{(n)}(0)}{n!}x^n+...$$

This case arises frequently enough and is given the special name Maclaurin series.

Example 3.14

Write the power series $x^4-5x^3+x^2-3x+4$ in terms of $(x-4)$.

Solution

Let $f(x)=x^4-5x^3+x^2-3x+4$,

First, we will find the first 4 derivatives at $x=4$;

$$f(x)=x^4-5x^3+x^2-3x+4 \Rightarrow f(4)=-56$$

$$f'(x)=4x^3-15x^2+2x-3 \Rightarrow f'(4)=21$$

$$f''(x)=12x^2-30x+2 \Rightarrow f''(4)=74$$

$$f'''(x)=24x-30 \Rightarrow f'''(4)=66$$

$$f^{iv}(x)=24$$

Therefore, power series of $f(x)$ is

$$56+21(x-4)+\frac{74}{2!}(x-4)^2+\frac{66}{3!}(x-4)^3+\frac{24}{4!}(x-4)^4$$

The power series in terms of $x-4$ is

$$f(x)=-56+21(x-4)+37(x-4)^2+11(x-4)^3+(x-4)^4$$

Example 3.15

Find the Taylor series of the function $f(x)=e^x$ at $x_0=2$

Solution

$$f(x)=e^x \Rightarrow f(0)=1$$

$$f'(x)=f''(x)=f'''(x)=f^{(4)}(x)=...=f^{(n)}(x)=e^x$$

$$\Rightarrow f'(2)=f''(2)=f'''(2)=f^{(4)}(2)...=f^{(n)}(2)=e^2$$

Then,

$$f(x)=e^x$$

$$=e^2+e^2(x-2)+\frac{e^2(x-2)^2}{2!}+\frac{e^2(x-2)^3}{3!}+\frac{(x-2)^4e^2}{4!}+\ldots+\frac{e^2(x-2)^n}{n!}+\ldots$$

$$=\sum_{n=0}^{\infty}\frac{e^2(x-2)^n}{n!}$$

Example 3.16

Find the Maclaurin series of order n for the function $f(x)=\frac{1}{1-x}$.

Solution

Order of derivative	Derivative	Value at $x=0$
0	$f(x)=\frac{1}{1-x}=(1-x)^{-1}$	1
1	$f'(x)=1(1-x)^{-2}$	1
2	$f''(x)=2\cdot1(1-x)^{-3}$	$2\cdot1=2!$
3	$f'''(x)=3\cdot2\cdot1(1-x)^{-4}$	$3\cdot2\cdot1=3!$
4	$f^{(4)}(x)=4\cdot3\cdot2\cdot1(1-x)^{-5}$	$4\cdot3\cdot2\cdot1=4!$
$\vdots$		
n	$f^{(n)}(x)=n\cdot(n-1)\ldots4\cdot3\cdot2\cdot1(1-x)^{-n-1}$	$n\cdot(n-1)\ldots4\cdot3\cdot2\cdot1=n!$

$$f(x)=1+x+\frac{2x^2}{2!}+\frac{3!x^3}{3!}+\frac{4!x^4}{4!}+\ldots+\frac{n!x^n}{n!}$$

$$f(x)=1+x+x^2+x^3+x^4+\ldots+x^n$$

$$=\sum_{k=0}^{n}x^k$$

Example 3.17

Find the Maclaurin series of the function $f(x)=(1+x)^m$ where $m\in\mathbb{R}$.

Solution

$$f(x)=(1+x)^m \Rightarrow f(0)=1$$

$$f'(x)=m(1+x)^{m-1} \Rightarrow f'(0)=m$$

$$f''(x)=m(m-1)(1+x)^{m-2} \Rightarrow f''(0)=m(m-1)$$

$$f'''(x)=m(m-1)(m-2)(1+x)^{m-3} \Rightarrow f'''(0)=m(m-1)(m-2)$$

$$\vdots$$

$$f^{(n)}(x)=m(m-1)(m-2)\dots(m-n+1)(1+x)^{m-n} \Rightarrow f^{(n)}(0)=m(m-1)(m-2)\dots(m-n+1)$$

Then,

$$f(x)=(1+x)^m$$

$$=1+mx+\frac{m(m-1)x^2}{2!}+\frac{m(m-1)(m-2)x^3}{3!}+\frac{m(m-1)(m-2)(m-3)x^4}{4!}+$$

$$\dots+\frac{m(m-1)(m-2)(m-3)\dots(m-n+1)x^n}{n!}+\dots$$

From which we can write;

$$(1+x)^m=1+\sum_{n=1}^{\infty}\frac{m(m-1)(m-2)(m-3)\dots(m-n+1)x^n}{n!}$$

Example 3.18

Find the Maclaurin series for the function $f(x)=\ln(1+x)$.

Solution

$$f(x)=\ln(1+x) \Rightarrow f(0)=0$$

$$f'(x)=\frac{1}{1+x} \Rightarrow f'(0)=1$$

$$f''(x)=\frac{-1}{(1+x)^2} \Rightarrow f''(0)=-1$$

$$f'''(x)=\frac{2!}{(1+x)^3}\Rightarrow f'''(0)=2!$$

$$\vdots$$

$$f^{(n)}(x)=\frac{(-1)^{n+1}(n-1)!}{(1+x)^n}\Rightarrow f^{(n)}(0)=(-1)^{n+1}(n-1)!$$

Thus, the required series is

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(n-1)!}{n!}x^n=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+(-1)^{n+1}\frac{x^n}{n}+\ldots$$

Application activity 3.28

Determine the Taylor series for each of the following functions at given value of a.

a) $f(x)=x-x^3$ at $a=-2$

b) $f(x)=\frac{1}{x}$ at $a=2$

c) $f(x)=e^{-2x}$ at $a=\frac{1}{2}$

d) $f(x)=\sin x$ at $a=\frac{\pi}{4}$

Taylor series by using Maclaurin series

Activity 3.6

1. Find the Maclaurin series for:

   a) $\sin x$ b) $\cos x$ c) $\ln(1+x)$

2. From the results in 1) or otherwise, find Maclaurin series for $\sin 2x$, $\cos 2x$ and $\ln(1+2x)$.

It is possible to find the Taylor series for other functions by using Maclaurin series $(x_0=0)$ without necessarily using Taylor's formula. These are some important Maclaurin series

1. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

2. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \cdots$

3. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$

4. If $-1 < x < 1$, then

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + \frac{m(m-1)(m-2)\cdots(m-n+1)}{n!}x^n + \cdots$$

Particularly, if $|x| < 1$, then we can write;

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + \ldots$$

Thus if $-1 < x \leq 1$, then $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{(-1)^{n-1} x^n}{n} + \ldots$

Example 3.19

Find the Taylor series for;

(i) $f(x) = e^{2x}$ at $x_0 = 0$ (ii) $f(x) = \ln x$ at $x_0 = 1$

Solution

(i) Let $2x = t$, and recall that $t \to 0$ as $x \to 0$. Using the series for e^x, we have

$$\begin{aligned} e^{2x} = e^t &= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots + \frac{t^n}{n!} + \ldots \\ &= 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \ldots + \frac{(2x)^n}{n!} + \ldots \\ &= 1 + 2x + \frac{4x^2}{2!} + \frac{8}{3!}x^3 + \ldots + \frac{2^n}{n!}x^n + \ldots \end{aligned}$$

Therefore,

$$e^{2x}=1+2x+2x^2+\frac{4}{3}x^3+\cdots+\frac{2^n}{n!}x^n+\cdots$$

(ii) $f(x)=\ln x=\ln(1+(x-1))$

Let $x-1=t$, since $t\to 0$ as $x\to 1$

Using the series $\ln(1+x)$ for the above, we have

$$\ln x=\ln(1+(x-1))\ =\ln(1+t)\ =t-\frac{t^2}{2}+\ldots+(-1)^{n-1}\frac{t^n}{n!}+\ldots$$

(Since $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+\ldots+\frac{(-1)^{n-1}x^n}{n}+\ldots$)

$$=(x-1)-\frac{(x-1)^2}{2}+\ldots+(-1)^{n-1}\frac{(x-1)^n}{n!}+\ldots$$

Hence, $f(x)=\ln x=(x-1)-\frac{(x-1)^2}{2}+\ldots+(-1)^{n-1}\frac{(x-1)^n}{n!}+\ldots$

Application activity 3.29

1. Write down Taylor series for each of the following functions at the given value of a.

   a) $f(x)=\frac{1}{x}$ at $x_0=3$

   b) $f(x)=\begin{cases}\frac{e^x-1}{x}, & x\neq 0\\ 1, & x=0\end{cases}$ at $x_0=0$

   c) $f(x)=\begin{cases}\frac{\sin x}{x}, & x\neq 0\\ 1, & x=0\end{cases}$, $x_0=0$

   d) $f(x)=\sin\frac{\pi x}{4}$ at $x_0=2$

2. Determine the Maclaurin series for each of the following functions:

   a) $f(x)=\cos^2 x$ b) $f(x)=x^2e^x$

   c) $f(x)=\sqrt{1-x^3}$ d) $f(x)=\frac{e^x}{1-x}$

3.3. Applications

Maclaurin series has several applications which include:

3.3.1. Calculation of limits

Activity 3.7

Suppose that we need to evaluate $\lim_{x\to 0}\frac{1-\cos 4x+x\sin 3x}{x^2}$

1. Find the Maclaurin series of order 3 for $\cos 4x$ and $\sin 3x$.
2. Replace the series obtained in 1) in the expression $\frac{1-\cos 4x+x\sin 3x}{x^2}$ and hence evaluate

$$\lim_{x\to 0}\frac{1-\cos 4x+x\sin 3x}{x^2}$$

The n^{th} order Maclaurin polynomial can help us to evaluate limits of some functions as illustrated in the following examples.

Example 3.20

Calculate; (i) $\lim_{x\to 0}\frac{\sin x-x}{x^3}$ (ii) $\lim_{x\to 0}\frac{e^{-\frac{x^2}{2}}-\cos x}{x^3\sin x}$

Solution

(i) For $n=3$, the Maclaurin series of $\sin x$ is $\sin x = x-\frac{x^3}{6}$

Then,

$$\lim_{x\to 0}\frac{\sin x-x}{x^3}=\lim_{x\to 0}\frac{x-\frac{x^3}{6}-x}{x^3}$$

$$=\lim_{x\to 0}\frac{-\frac{x^3}{6}}{x^3}=\lim_{x\to 0}\frac{-\frac{x^3}{6}}{x^3}=-\lim_{x\to\infty}\frac{x^3}{6x^3}=-\frac{1}{6}$$

(ii) For $n=4$

$$e^{x}=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\frac{x^{4}}{4!}$$

$$e^{-\frac{x^{2}}{2}}=1+\left(-\frac{x^{2}}{2}\right)+\frac{\left(-\frac{x^{2}}{2}\right)^{2}}{2!}+\frac{\left(-\frac{x^{2}}{2}\right)^{3}}{3!}+\frac{\left(-\frac{x^{2}}{2}\right)^{4}}{4!}$$

$$e^{-\frac{x^{2}}{2}}=1-\frac{x^{2}}{2}+\frac{x^{4}}{4\cdot 2!}-\frac{x^{6}}{8\cdot 3!}+\frac{x^{8}}{16\cdot 4!}$$

$$\sin x=x-\frac{x^{3}}{6}$$

$$\cos x=1-\frac{x^{2}}{2}+\frac{x^{4}}{24}$$

$$\lim_{x\to 0}\frac{e^{-\frac{x^{2}}{2}}-\cos x}{x^{3}\sin x}=\lim_{x\to 0}\frac{1-\frac{x^{2}}{2}+\frac{x^{4}}{4\cdot 2!}-\frac{x^{6}}{8\cdot 3!}+\frac{x^{8}}{16\cdot 4!}-\left(1-\frac{x^{2}}{2}+\frac{x^{4}}{24}\right)}{x^{3}\left(x-\frac{x^{3}}{6}\right)}$$

$$=\lim_{x\to 0}\frac{1-\frac{x^{2}}{2}+\frac{x^{4}}{4\cdot 2!}-\frac{x^{6}}{8\cdot 3!}+\frac{x^{8}}{16\cdot 4!}-1+\frac{x^{2}}{2}-\frac{x^{4}}{24}}{x^{3}\left(x-\frac{x^{3}}{6}\right)}$$

$$=\lim_{x\to 0}\frac{\frac{x^{4}}{8}-\frac{x^{6}}{48}+\frac{x^{8}}{384}-\frac{x^{4}}{24}}{x^{3}\left(x-\frac{x^{3}}{6}\right)}=\lim_{x\to 0}\frac{x^{4}\left(\frac{1}{8}-\frac{x^{2}}{48}+\frac{x^{4}}{384}-\frac{1}{24}\right)}{x^{4}\left(1-\frac{x^{2}}{6}\right)}$$

$$=\lim_{x\to 0}\frac{\frac{1}{8}-\frac{x^{2}}{48}+\frac{x^{4}}{384}-\frac{1}{24}}{1-\frac{x^{2}}{6}}=\frac{1}{8}-\frac{1}{24}=\frac{3-1}{24}=\frac{1}{12}$$

Application activity 3.30

Using Maclaurin series, evaluate:

1. $\lim_{x\to 0}\left(\frac{\ln(1+x)-x}{\sin^2 x}\right)$ 2. $\lim_{x\to 0}\left(\frac{-x+\tan x}{x-\sin x}\right)$

3. $\lim_{x\to 0}\frac{e^x-1-x}{x^2}$ 4. $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$

3.3.2. Estimation of the number e

Activity 3.8

In Maclaurin series of order n for e^x, replace x with 1 and hence estimate the value of number e to 8 decimal places.

By putting $x=1$ in the development of e, we can easily estimate the value of the number e to desired decimal places.

Example 3.21

Estimate the value of number e to 3 decimal places.

Solution

In series $e=1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+...+\frac{1}{n!}$, the general term is $\frac{1}{n!}$. Since we need number e to 3 decimal places, we need to find the smallest value of n first such that

$$\left|\frac{1}{(n+1)!}\right|<0.001 \text{ or } 1<0.001(n+1)!$$

Here, $n=6$ since $1<0.001(7!)\Leftrightarrow 1<5.04$

Then,

$$\begin{aligned} e &\approx 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!} \\ &\approx 2+0.5+0.167+0.042+0.008+0.001 \\ &\approx 2.718 \end{aligned}$$

Example 3.22

Estimate the value of number e to 5 decimal places.

Solution

In series $e=1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+...+\frac{1}{n!}$, the general term is $\frac{1}{n!}$.

Since we need number e to 5 decimal places,

we need first to find the smallest value of n such that

$$\left|\frac{1}{(n+1)!}\right|<0.00001 \text{ or } 1<0.00001(n+1)!.$$

Here, $n=8$ since $1<0.00001(9!)\Leftrightarrow 1<3.6288$

Then,

$$\begin{aligned} e &\approx 1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\frac{1}{8!} \\ &\approx 2+0.5+0.16667+0.04167+0.00833+0.00139+0.00019+0.00002 \\ &\approx 2.71827 \end{aligned}$$

Application activity 3.31

Estimate the value of number e to:

1. 2 decimal places
2. 4 decimal places
3. 6 decimal places
4. 10 decimal places

3.3.3. Estimation of the number π

Activity 3.9

1. Find the Maclaurin series for $\arctan x$.
2. Solve $\tan x=\frac{\sqrt{3}}{3},\ 0<x<\frac{\pi}{2}$.

From activity 3.9,

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + (-1)^n \frac{x^{2n+1}}{2n+1} + \ldots$$

Also, $\frac{\pi}{6} = \arctan \frac{\sqrt{3}}{3}$

Then by setting $\frac{\sqrt{3}}{3}$ in (1) we have;

$$\frac{\pi}{6} = \frac{\sqrt{3}}{3} - \frac{\left(\frac{\sqrt{3}}{3}\right)^3}{3} + \frac{\left(\frac{\sqrt{3}}{3}\right)^5}{5} - \frac{\left(\frac{\sqrt{3}}{3}\right)^7}{7} + \ldots + (-1)^n \frac{\left(\frac{\sqrt{3}}{3}\right)^{2n+1}}{2n+1} + \ldots$$

Or

$$\pi = 6\frac{\sqrt{3}}{3} - 6\frac{\left(\frac{\sqrt{3}}{3}\right)^3}{3} + 6\frac{\left(\frac{\sqrt{3}}{3}\right)^5}{5} - 6\frac{\left(\frac{\sqrt{3}}{3}\right)^7}{7} + \ldots + (-1)^n 6\frac{\left(\frac{\sqrt{3}}{3}\right)^{2n+1}}{2n+1} + \ldots$$

Example 3.23

Estimate the value of number π to 2 decimal places.

Solution

$$\pi = 6\frac{\sqrt{3}}{3} - 6\frac{\left(\frac{\sqrt{3}}{3}\right)^3}{3} + 6\frac{\left(\frac{\sqrt{3}}{3}\right)^5}{5} - 6\frac{\left(\frac{\sqrt{3}}{3}\right)^7}{7} + \ldots + (-1)^n 6\frac{\left(\frac{\sqrt{3}}{3}\right)^{2n+1}}{2n+1} + \ldots$$

The general term is $(-1)^n \frac{\left(\frac{\sqrt{3}}{3}\right)^{2n+1}}{2n+1}$

We need n such that

$$\left| \frac{\left(\frac{\sqrt{3}}{3}\right)^{2(n+1)+1}}{2(n+1)+1} \right| = \left| \frac{\left(\frac{\sqrt{3}}{3}\right)^{2n+3}}{2n+3} \right| < 0.01 \quad \text{since we need two decimal places.}$$

Here, $n=2$ since $\left|\dfrac{\left(\frac{\sqrt{3}}{3}\right)^7}{7}\right| = 0.003 < 0.01$

Then,

$$\pi = 6\frac{\sqrt{3}}{3} - 6\frac{\left(\frac{\sqrt{3}}{3}\right)^3}{3} + 6\frac{\left(\frac{\sqrt{3}}{3}\right)^5}{5} = 3.46 - 0.38 + 0.07$$
$$= 3.15$$

Application activity 3.32

Estimate the value of number e to:

1. 3 decimal places
2. 5 decimal places
3. 7 decimal places
4. 9 decimal places

3.3.4. Estimation of trigonometric number of an angle

Activity 3.10

1. Find the Maclaurin series of order n for $\sin x$.
2. By letting $x = \dfrac{\pi}{4}$ in 1), estimate the value of $\sin\dfrac{\pi}{4}$ to 4 decimal places.

x being expressed in radians, we can approximate the value of any trigonometric number using the series of trigonometric functions.

Example 3.24

Estimate the value of $\cos\dfrac{\pi}{6}$ to 3 decimal places.

Solution

We first need to obtain Maclaurin series for $f(x) = \cos x$ where $x = \dfrac{\pi}{6}$.

The Maclaurin series of order n for $f(x) = \cos x$:

$$f(x)=\cos x \Rightarrow f(0)=1$$

$$f'(x)=-\sin x \Rightarrow f'(0)=0$$

$$f''(x)=-\cos x \Rightarrow f''(0)=-1$$

$$f'''(x)=\sin x \Rightarrow f(0)=0$$

$$f^{(4)}(x)=\cos x \Rightarrow f^{(4)}(0)=1$$

$$\cos x = 1+\frac{0x}{1!}-\frac{x^2}{2!}+\frac{0x^3}{3!}+\frac{x^4}{4!}+\ldots+(-1)^n\frac{x^{2n}}{(2n)!}$$

Or

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots+(-1)^n\frac{x^{2n}}{(2n)!}$$

Putting $x=\frac{\pi}{6}$, we have

$$\cos\frac{\pi}{6}=1-\frac{\left(\frac{\pi}{6}\right)^2}{2!}+\frac{\left(\frac{\pi}{6}\right)^4}{4!}+\ldots+(-1)^n\frac{\left(\frac{\pi}{6}\right)^{2n}}{(2n)!}$$

The general term is $(-1)^n\frac{\left(\frac{\pi}{6}\right)^{2n}}{(2n)!}$. Since we need the value of $\cos\frac{\pi}{6}$ to 3 decimal places, we need the value of n such that

$$\left|\frac{\left(\frac{\pi}{6}\right)^{2(n+1)}}{(2(n+1))!}\right|<0.001$$

Here, $n=2$ because $\frac{\left(\frac{\pi}{6}\right)^{2(2+1)}}{(2(2+1))!}=0.00002<0.001$

Thus,

$$\cos\frac{\pi}{6} \approx 1 - \frac{\left(\frac{\pi}{6}\right)^2}{2!} + \frac{\left(\frac{\pi}{6}\right)^4}{4!} \approx 1 - 0.137 + 0.003 \approx 0.866$$

Application activity 3.33

Using Maclaurin series, estimate:

1. The number $\sin 1^0$ to 6 decimal places.
2. The number $\sin\frac{2\pi}{3}$ to 3 decimal places.
3. The number $\cos 65^0$ to 4 decimal places.
4. The number $\cos\left(-135^0\right)$ to 5 decimal places.

3.3.5. Estimation of an irrational number

Activity 3.11

Suppose that we need to estimate the value of $\sqrt{2}$ to 6 decimal places.

1. Write down the squares of natural numbers (as we need square root).
2. Multiply each term in the obtained sequence in 1) by the radicand (here radicand is 2).
3. Take two numbers from sequence in 1) and another from sequence in 2) such that their ratio is close to 1.
4. Using the obtained numbers from 3), transform the radicand so that it differs a little from 1.

Using the Maclaurin series of $(1+x)^m$, we can estimate any irrational number like $\sqrt{2}, \sqrt{3}, \sqrt[3]{5}, \ldots$

Example 3.25

Estimate the value of $\sqrt{2}$ to 6 decimal places.

Solution

From activity 3.11, we can take the numbers 49 and 50. Note that when extending the series, we can find other numbers such that their ratio is closed to 1.

Then,

$$\sqrt{2}=\sqrt{\frac{2\times 25\times 49}{25\times 49}}=\frac{7}{5}\sqrt{\frac{2\times 25}{49}}=\frac{7}{5}\sqrt{\frac{50}{49}}=\frac{7}{5}\sqrt{1+\frac{1}{49}}=\frac{7}{5}\left(1+\frac{1}{49}\right)^{\frac{1}{2}}$$

Now, recall that the Maclaurin series of $(1+x)^m$ is

$$(1+x)^m=1+\sum_{n=1}^{\infty}\frac{m(m-1)(m-2)(m-3)\dots(m-n+1)x^n}{n!}$$

Putting $x=\frac{1}{49}$ and $m=\frac{1}{2}$

$$\left(1+\frac{1}{49}\right)^{\frac{1}{2}}=1+\sum_{n=1}^{\infty}\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\dots\left(\frac{1}{2}-n+1\right)\left(\frac{1}{49}\right)^n}{n!}$$

The general term is $\dfrac{\left(\frac{1}{2}-n+1\right)\left(\frac{1}{49}\right)^n}{n!}=\dfrac{\left(\frac{3}{2}-n\right)\left(\frac{1}{49}\right)^n}{n!}$

Since we need the value of $\sqrt{2}$ to 6 decimal places, we need the value of n such that

$$\left|\frac{\left(\frac{3}{2}-(n+1)\right)\left(\frac{1}{49}\right)^{n+1}}{(n+1)!}\right|<0.000001$$

Here, $n=3$ because $\left|\dfrac{\left(\frac{3}{2}-(4)\right)\left(\frac{1}{49}\right)^4}{4!}\right|=0.00000002<0.000001$

Then,

$$\sqrt{2} \approx \frac{7}{5}\left(1+\frac{1}{49}\right)^{\frac{1}{2}} \approx \frac{7}{5}\left(1+\frac{\frac{1}{2}\left(\frac{1}{49}\right)^{1}}{1!}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{49}\right)^{2}}{2!}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{49}\right)^{3}}{3!}\right)$$

$$\approx \frac{7}{5}(1+0.010204-0.000052+0.000005) \approx 1.4142198$$

Example 3.26

Estimate the value of $\sqrt[3]{5}$ to 2 decimal places.

Solution

In this case, write down cube of natural numbers starting from 1 as below:

1; 8; 27; 64; 125; 216; … (1)

Now, multiplying each term in sequence by 5 we obtain:

5; 40; 135; 320; 625; 1080; … (2)

Taking two numbers from sequences (1) and (2) such that their ratio is close to 1 i.e. 125 and 135

Then,

$$\sqrt[3]{5} = \sqrt[3]{\frac{5\times 27\times 125}{27\times 125}} = \frac{5}{3}\sqrt[3]{\frac{5\times 27}{125}} = \frac{5}{3}\sqrt[3]{1+\frac{10}{125}}$$

$$= \frac{5}{3}\left(1+\frac{10}{125}\right)^{\frac{1}{3}} = \frac{5}{3}\left(1+\frac{2}{25}\right)^{\frac{1}{3}}$$

Now, using Maclaurin series for $(1+x)^m$ i.e.

$$(1+x)^m = 1+\sum_{n=1}^{\infty}\frac{m(m-1)(m-2)(m-3)\ldots(m-n+1)x^n}{n!}$$

Putting $x=\frac{2}{25}$ and $m=\frac{1}{3}$ we have;

$$\left(1+\frac{2}{25}\right)^{\frac{1}{3}} = 1+\sum_{n=1}^{\infty}\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)\left(\frac{1}{3}-3\right)\ldots\left(\frac{1}{3}-n+1\right)\left(\frac{2}{25}\right)^{n}}{n!}$$

The general term is $\dfrac{\left(\frac{1}{3}-n+1\right)\left(\frac{2}{25}\right)^n}{n!}=\dfrac{\left(\frac{4}{3}-n\right)\left(\frac{2}{25}\right)^n}{n!}$

Since we need the value of $\sqrt[3]{5}$ to 2 decimal places, we need the value of n such that

$$\left|\frac{\left(\frac{4}{3}-(n+1)\right)\left(\frac{2}{25}\right)^{n+1}}{(n+1)!}\right|<0.01$$

Here, $n=1$ because $\left|\dfrac{\left(\frac{4}{3}-1\right)\left(\frac{2}{25}\right)^2}{2!}\right|=0.002<0.01$

Then,

$$\sqrt[3]{5}\approx\frac{5}{3}\left(1+\frac{2}{25}\right)^{\frac{1}{3}}\approx\frac{5}{3}\left(1+\frac{\frac{1}{3}\left(\frac{2}{25}\right)^1}{1!}\right)\approx\frac{5}{3}(1+0.03)\approx 1.71$$

Application activity 3.34

Estimate:

1. $\sqrt{3}$ to 3 decimal places.
2. $\sqrt{5}$ to 4 decimal places.
3. $\sqrt[3]{2}$ to 6 decimal places.
4. $\sqrt[3]{4}$ to 6 decimal places.

3.3.6. Estimation of natural logarithm of a number

Activity 3.12

1. Find the Maclaurin series of order n for $\ln(1+x)$.
2. In the result obtained in 1), replace x with $-x$ to obtain the expansion series for $\ln(1-x)$.
3. Subtract the result obtained in 2) from the result obtained in 1) to find the expansion for $\ln\frac{1+x}{1-x}$

Remember that $\ln x - \ln y = \ln\frac{x}{y}$

From activity 3.12,

$$\ln\frac{1+x}{1-x} = 2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots+\frac{x^{2n+1}}{2n+1}+\ldots\right) = 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}$$

This relation helps us to estimate ln of any positive number.

Example 3.27

Estimate: (i) $\ln 2$ to 4 decimal places.

(ii) $\ln 6$ to 3 decimal places.

Solution

(i) Let $\ln\frac{1+x}{1-x} = \ln 2 \Leftrightarrow \frac{1+x}{1-x} = 2 \Rightarrow x = \frac{1}{3}$

Then,

$$\ln 2 = 2\left(\frac{1}{3}+\frac{\left(\frac{1}{3}\right)^3}{3}+\frac{\left(\frac{1}{3}\right)^5}{5}+\ldots+\frac{\left(\frac{1}{3}\right)^{2n+1}}{2n+1}+\ldots\right)$$

The general term is $\frac{\left(\frac{1}{3}\right)^{2n+1}}{2n+1}$. Since we need the value of

$\ln 2$ to 4 decimal places, we need the value of n such that $\frac{\left(\frac{1}{3}\right)^{2(n+1)+1}}{2(n+1)+1} < 0.0001$

Thus, $n = 2$ because $\frac{\left(\frac{1}{3}\right)^{2(2+1)+1}}{2(2+1)+1} = 0.00006 < 0.0001$

Then,

$$\ln 2 \approx 2\left(\frac{1}{3} + \frac{\left(\frac{1}{3}\right)^3}{3} + \frac{\left(\frac{1}{3}\right)^5}{5}\right) \approx 2(0.3333 + 0.0123 + 0.0008) \approx 0.6928$$

(ii) Let $\ln\frac{1+x}{1-x} = \ln 6 \Leftrightarrow \frac{1+x}{1-x} = 6 \Rightarrow x = \frac{5}{7}$

Then,

$$\ln 6 = 2\left(\frac{1}{3} + \frac{\left(\frac{5}{7}\right)^3}{3} + \frac{\left(\frac{5}{7}\right)^5}{5} + \ldots + \frac{\left(\frac{5}{7}\right)^{2n+1}}{2n+1} + \ldots\right)$$

The general term is $\frac{\left(\frac{5}{7}\right)^{2n+1}}{2n+1}$. Since we need the value of

$\ln 6$ to 3 decimal places, we need the value of n such that $\frac{\left(\frac{1}{3}\right)^{2(n+1)+1}}{2(n+1)+1} < 0.001$

Thus, $n = 5$ because $\frac{\left(\frac{5}{7}\right)^{2(2+1)+1}}{2(2+1)+1} = 0.0009 < 0.001$

Then,

$$\ln 6 \approx 2\left(\frac{5}{7}+\frac{\left(\frac{5}{7}\right)^3}{3}+\frac{\left(\frac{5}{7}\right)^5}{5}+\frac{\left(\frac{5}{7}\right)^7}{7}+\frac{\left(\frac{5}{7}\right)^9}{9}+\frac{\left(\frac{5}{7}\right)^{11}}{11}\right)$$

$$\approx 2(0.714+0.121+0.037+0.014+0.005+0.002)\approx 1.786$$

Application activity 3.35

Using Maclaurin series, estimate:

1. $\ln 3$ to 4 decimal places.
2. $\ln 0.8$ to 3 decimal places.
3. $\ln 7$ to 5 decimal places.
4. $\ln 0.2$ to 2 decimal places.

3.3.7. Estimation of roots of equations

Activity 3.13

Consider the equation $\ln(1+x)+x=0$

1. Find the second order Maclaurin polynomial of $\ln(1+x)$.
2. Put the result obtained in 1) in the given equation.
3. Solve for x in the new equation obtained in 2).
4. Check if the value(s) of x obtained in 3) satisfies the given equation, $\ln(1+x)+x=0$ and hence write down the solution set.

From activity 3.13, the n^{th} order Maclaurin polynomial can help us to estimate the roots of a given equation involving transcendental functions.

Example 3.28

Solve in $\mathbb{R}$, the equation $\ln(1-x)+e^x=1$

Solution

Maclaurin polynomial of $\ln(1-x)$ is $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots$

Maclaurin polynomial of e^x is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots$

Now,

$$\ln(1-x) + e^x = 1 \Leftrightarrow -x - \frac{x^2}{2} - \frac{x^3}{3} + 1 + x + \frac{x^2}{2} + \frac{x^3}{6} = 1$$

$$\Leftrightarrow -\frac{x^3}{3} + 1 + \frac{x^3}{6} = 1 \Leftrightarrow \frac{-2x^3 + x^3}{6} = 0 \Leftrightarrow -x^3 = 0 \Rightarrow x = 0$$

Check if this is a root of the given equation:

LHS= $\ln(1-0) + e^0 = (\ln 1) + 1 = 1$ and RHS=1

Since LHS=RHS=1; x=0

Therefore, $S = \{0\}$

Application activity 3.36

Using Maclaurin polynomial, estimate the roots of the following equation in $\mathbb{R}$:

1. $\cos x - 2x^2 = 0$
2. $x - e^{-x} = 0$
3. $\ln(x^2 + 3x + 1) = x$
4. $e^{\sqrt{x}} - 3x = -5$

Unit summary

1. Generalities on series

- **Definitions**

A finite series is an expression of the form $u_1 + u_2 + u_3 + ... + u_n$ or in sigma notation $\sum_{k=1}^{n} u_k$,

where the index of summation, k, takes consecutive integer values from the lower limit, 1, to the upper limit, n. The terms $u_1, u_2, u_3, ..., u_n$ are called **terms** of the series and the term u_n is the general term.

To obtain $\sum_{k=1}^{n} u_n$, the method of **difference** is usually used i.e.

$\sum_{k=1}^{n} u_n = f(1) - f(n+1)$ where $u_k = f(k) - f(k+1)$,
with $f(k)$ some function of k.

- **Convergence and divergence of a series**

Let $\{s_n\}$ be the sequence of partial sums of the series $\sum_{k=1}^{+\infty} u_k$. If the sequence $\{s_n\}$ converges to a limit S, then the series is said to converge and S is called the **sum** of the series. We denote this by writing $S = \sum_{k=1}^{+\infty} u_k$.

If the sequence of partial sums of a series diverges, then the series is said to diverge. A divergent series has no sum.

- **Comparison test**

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms.

a) $\sum_{n=1}^{\infty} a_n$ converges if there exists a convergent series $\sum_{n=1}^{\infty} b_n$ such that $a_n \le b_n$ for all $n > N$, where N is some positive integer.

b) $\sum_{n=1}^{\infty} a_n$ diverges if there exists a divergent series $\sum_{n=1}^{\infty} c_n$ such that $a_n \ge c_n$ for all $n > N$, where N is some positive integer.

- **Limit comparison test**

  If the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are two series with positive terms, and $\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite, both series converges or both diverges.

- **The ratio test**

  Let $\sum_{n=1}^{\infty} u_n$ be a series with positive terms and let $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L$, then;

  a) the series converges if $L < 1$,

  b) the series diverges if $L > 1$,

  c) the series may or may not converge if $L = 1$ (i.e., the test is inconclusive).

- **The n^{th} root test**

  Let $\sum_{n=1}^{\infty} u_n$ be a series with positive terms and let $\lim_{n\to\infty} \sqrt[n]{u_n} = L$, then,

  a) the series converges if $L < 1$

  b) the series diverges if $L > 1$

  c) the test is inconclusive if $L = 1$.

2. Power series

Power series is like an infinite polynomial. It has the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots + a_n(x-c)^n + \ldots$$

Here c is any real number and a series of this form is called a **power series centred at c**.

Let $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ be the function defined by this power series. $f(x)$ is only defined if the power series converges, so we will consider the domain of the function f to be the set of x values for which the series converges. There are **three** possible cases:

» The power series converges at $x = c$. Here the radius of convergence is zero.

» The power series converges for all x, i.e $]-\infty, +\infty[$. Here, the radius of convergence is infinity.

» There is a number R called the radius of convergence such that the series converges for all $c - R < x < c + R$ and the series diverges outside this interval.

3. Taylor and Maclaurin series

If $f(x)$ is a function defined on the open interval (a, b), and which can be differentiated $(n+1)$ times on (a, b), then the equality

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_{n+1}(x)$$

for any values of x and x_0 in (a, b) is called **Taylor's formula**.

$R_{n+1}(x)$ is called the **remainder function**.

The resulting function (without $R_{n+1}(x)$) is called the **Taylor expansion** of $f(x)$ with respect to x about the point $x = x_0$ of order n.

One of the most common forms of the **remainder** function is the Lagrange form:

$$R_{n+1}(x) = \frac{(x - x_0)^{n+1}}{(n+1)!} f^{(n+1)}\left(x_0 + \theta(x - x_0)\right)$$

where $0 < \theta < 1$.

If $\lim\limits_{n \to \infty} R_{n+1}(x) = 0$ for some terms, then the infinite series

$$f(x) = f(x_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

is called the Taylor series for $f(x)$.

A Maclaurin series is a **Taylor series** with $x_0 = 0$.

Note that if $f(x)$ is a polynomial of degree n, then it will have utmost only n non-zero derivatives; all other higher-order derivatives will be identically equal to zero.

The following series are very important. All of them are Maclaurin series $(x_0 = 0)$ and, it is possible to find the Taylor series for other functions by using these formulae without necessarily using Taylor's formula.

a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

b) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots$

c) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n \frac{x^{2n}}{(2n)!}$

d) If $-1 < x < 1$, then

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)x^3}{3!} + \cdots$$

$$+ \frac{m(m-1)(m-2)\ldots(m-n+1)x^n}{n!} + \cdots$$

Particularly, if $|x| < 1$, then

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + \ldots$$

If $-1 < x \le 1$, then $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{(-1)^{n-1} x^n}{n} + \ldots$

End of unit assessment

1. In questions a-e, sum the given series:

a) $\sum_{r=1}^{n} r(r+4)$

b) $\sum_{r=1}^{n} (r+2)^3$

c) $\sum_{r=1}^{n} \frac{1}{(r+3)(r+6)}$

d) $\sum_{r=1}^{n} r(2+r)$

e) $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$

2. In questions a-c find the rational number represented by the repeating decimal.

a) $0.2\overline{35}$ b) $0.\overline{50}$ c) $0.\overline{011}$

3. In questions a-f, determine both the radius of convergence and the interval of convergence.

a) $\sum_{n=0}^{\infty} \frac{(2x+3)^n}{4^n}$

b) $\sum_{n=1}^{\infty} n^3 x^n$

c) $\sum_{n=0}^{\infty} \frac{x^n}{n^2+2}$

d) $\sum_{n=0}^{\infty} \frac{n^2 x^n}{2^n}$

e) $\sum_{n=1}^{\infty} \frac{x^n}{2^n \sqrt{n}}$

f) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2x)^n}{n 3^n}$

4. Write down the first 4 terms of the Taylor series for the following functions:

a) $\ln x$ centred at $a=1$

b) $\frac{1}{x}$ centred at $a=1$

c) $\sin x$ centred at $a=\frac{\pi}{4}$

5. Determine the first three terms of the Taylor series for the function $\sin \pi x$ centred at $a=\frac{1}{2}$. Use your answer to find an approximate value to $\sin\left(\frac{\pi}{2}+\frac{\pi}{10}\right)$.

6. Determine the Taylor series for the function $x^4 + x - 2$ centred at $a = 1$.
7. Obtain the Taylor series for $(x-1)e^x$ near $x = 1$.
8. Write down the first three terms in the Maclaurin series for:

   a) $\sin^2 x$ b) $\dfrac{x}{\sqrt{1-x^2}}$ c) xe^{-x} d) $\dfrac{x}{1+x^2}$

9. Determine the Maclaurin series for $\ln(1+x)$ and hence that for $\ln\dfrac{1+x}{1-x}$.
10. If function $\ln(1+x)$ is approximated by the first three terms of its Maclaurin series, estimate the maximum value of x for which the approximation agrees with the exact value to 3 decimal places.
11. By using a suitable Maclaurin series in the text, find the sum to infinity of the following series:

    a) $\pi - \dfrac{\pi^3}{3!} + \dfrac{\pi^5}{5!} - \dfrac{\pi^7}{7!} + \ldots$ b) $1 - \dfrac{e^2}{2!} + \dfrac{e^4}{4!} - \dfrac{e^6}{6!} + \ldots$

12. Determine the Maclaurin series for $x \sin x$.
13. The kinetic energy of a relativistic particle is given by $K = (\gamma - 1)mc^2$ where $\gamma = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$. Here, m is the constant mass of the particle, v its speed and c is the constant speed of light. Use the Maclaurin series for $\dfrac{1}{\sqrt{1-x^2}}$ to show that for $v < c,\ K = \dfrac{1}{2}mv^2$.
14. Obtain the first three terms in the Maclaurin series for $\cos(\sin x)$. Hence or otherwise, evaluate $\lim\limits_{x\to 0}\dfrac{1-\cos(\sin x)}{x^2}$.
15. Determine the first three terms in the Maclaurin series for $\sin(\sin x)$. Hence or otherwise, find $\lim\limits_{x\to 0}\dfrac{x-\sin(\sin x)}{x^3}$.
16. The equation $e^{-2x} = 3x^2$ has a root near $x = 0$. By using a suitable polynomial approximation to e^{-2x}, obtain an approximation to this root.

17. Write down the Maclaurin series for the function $f(x)=\ln(1+x)$ and hence, obtain the series for $f(x)=\dfrac{\ln(1+x)}{x}$.

18. Determine the first 3 terms in the Maclaurin series for $\sqrt{1-x+x^2}$

19. Write down the Maclaurin series for the function $\dfrac{1}{1-x^2}$ by using partial fractions or otherwise.

20. Determine the Maclaurin series for the function $\dfrac{x+1}{x^2-5x+6}$ by first finding the partial fraction

    decomposition of the function.

21. Obtain the first three non-zero terms of the Maclaurin series for $f(x)=e^{-x^2}\sin x$. Hence or otherwise,

    evaluate $\lim\limits_{x\to 0}\dfrac{f(x)-x}{x^3}$.

22. Determine the Maclaurin series for the functions e^x and $\sin x$, and hence expand $e^{\sin x}$ up to the term in x^4.

23. The Maclaurin series for e^z converges for all z including the case when z is a complex number. Using this fact, write down the Maclaurin series for $e^{i\theta}$ and hence prove Euler's formula $e^{i\theta}=\cos\theta+i\sin\theta$. Also, deduce the extraordinary relation $e^{i\pi}+1=0$.

24. Consider the infinite series $\sum\limits_{n=1}^{\infty}\left(\sqrt[3]{n^3+1}-n\right)$.

    a) Give the first three terms of the Maclaurin expansion of the function $f(x)=\sqrt[3]{1+x}$.

    b) Use your result in a) to show that for a large n, the general term of the given infinite series behaves as $\dfrac{1}{3n^2}$

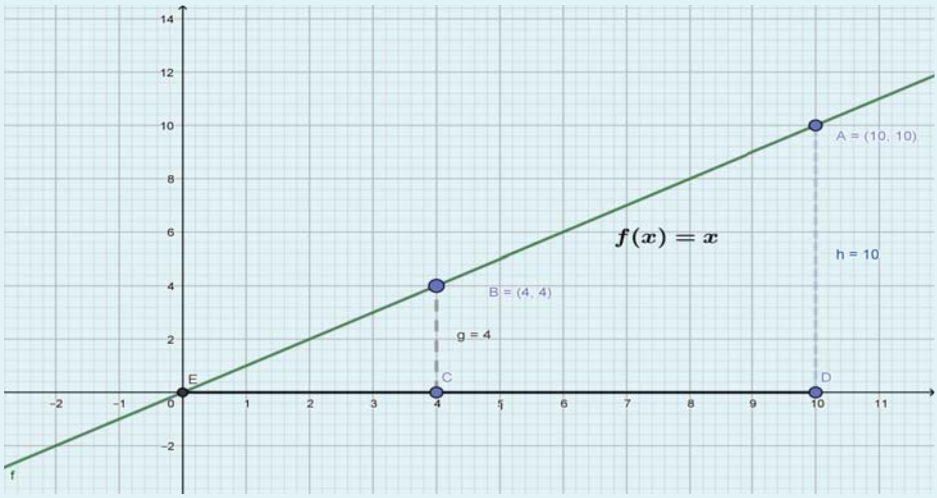
    c) Hence, show that the given infinite series converges.

Unit 4

Integration

Introductory activity

Two groups of students were asked to calculate the area of a quadrilateral field BCDA shown in the following figure:



The first group calculated the difference of the area for two triangles EDA and ECB

$A_1 = area(\Delta EDA) - area(\Delta ECB)$, The second group with high critical thinking skills used a function $F(x)$ that was differentiated to find $f(x) = x$ (which means $F'(x) = f(x)$ and the x-coordinate d of D and the x- coordinate c of C in the following way: $A_2 = F(d) - F(c)$.

1. Determine the area A_1 found by the first group.
2. Discuss and determine the function $F(x)$ used by the second group. What is the name of $F(x)$ if you relate it with $f(x)$?
3. Determine A_2 he area found by the second group using $F(x)$
4. Compare A_1 and A_2. Discuss if it is possible to find the area bounded by a function $f(x)$, the x-axis and lines with equation $x = x_1$ and $x = x_2$?

Objectives

By the end of this unit, I will be able to:

- Define the differential of a function.
- Interpret geometrically the differential of a function.
- List the differentiation formulae.
- Clarify the relationship between derivative and anti-derivative of a function.
- Illustrate the use of basic integration formulae.
- Extend the concepts of indefinite integrals to definite integrals.
- Use integrals to find area of a plane surfaces, volume of a solid of revolution and length of curved lines.

4.1. Differentials

Activity 4.1

Without using scientific calculator, determine;

a) approximately how much the value of $\sin x$ increases from $\frac{\pi}{3}$ to $\frac{\pi}{3}+0.006$.

b) to 3 decimal places the value of $\sin\left(\frac{\pi}{3}+0.006\right)$.

If one quantity, say y, is a function of another quantity x, that is, $y=f\left(x\right)$, we sometimes want to know how a change in the value of x by an amount Δx will affect the value of y.

The variation in x is Δx and it is called increment of x while the corresponding variation in y becomes $\Delta y=f(x+\Delta x)-f(x)$.

The increment of $y=f(x)$ is $\Delta y=f(x+\Delta x)-f(x)$. This can be found in the other way.

The exact change, Δy in y, is given by $\Delta y = f(x+\Delta x) - f(x)$ as it is shown in figure 4.1.

But if the change Δx is small, then we can get a good

approximation to Δy by using the fact that $\frac{\Delta y}{\Delta x}$ is approximately the derivative $\frac{dy}{dx}$. Thus, we can write

$$\Delta y = \frac{\Delta y}{\Delta x}\Delta x \approx \frac{dy}{dx}\Delta x = f'(x)\Delta x$$

If we denote the change in x by δx instead of Δx, then the change, Δy in y, is approximated by the differential δy, that is $\Delta y \approx \delta y \approx f'(x)dx$.

The differential of a function $f(x)$ is the approximated increment of that function when the variation in x becomes very small. It is given by $dy = f'(x)dx$.

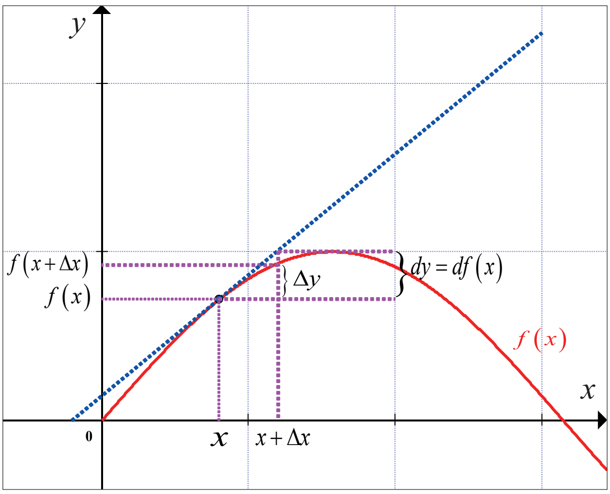


***Figure 4.1.** Change in the value of x and y*

Example 4.1

Find the differential of $f(x) = x^2 + 1$

Solution

$f'(x) = 2x$

Then the differential of $f(x) = x^2 + 1$ is $d[f(x)] = f'(x)dx = 2xdx$

Notice

Whenever one makes an approximation, it is wise to try and estimate how big the error might be.

- Relative change in x is $\frac{\Delta x}{x}$.
- Percentage change in x is $100 \times \frac{\Delta x}{x}$.

Example 4.2

By approximately what percentage does the area of a circle increase if the radius increases by 2%?

Solution

The area A of a circle is given in terms of the radius r by $A = \pi r^2$

Now; $\Delta A = \delta A \approx \frac{dA}{dr}\delta r = 2\pi r \delta r$

Dividing this approximation by $A = \pi r^2$ gives an approximation that links the relative changes in A and r:

$$\frac{\Delta A}{A} = \frac{\delta A}{A} \approx \frac{2\pi r \delta r}{\pi r^2} = 2\frac{\delta r}{r}$$

If r increases by 2%, then $\delta r = \frac{2}{100}r$, so $\frac{\Delta A}{A} \approx 2 \times \frac{2}{100} = \frac{4}{100}$

Thus, A increases by approximately 4%.

Example 4.3

The deflection at the centre of a road of length l and diameter d supported at its ends and loaded at the centre with a weight w varies as wl^3d^{-4}. What is the percentage increase in the deflection corresponding to the percentage increase in w, l and d of $3, 2$ and 1 respectively?

Solution

Let the deflection of the road at the centre be D.

$$\Rightarrow D = k\frac{wl^3}{d^4}$$

Introducing natural logarithm on both sides of the expression we have;

$$\ln D = \ln k \frac{wl^3}{d^4}$$

$$\Rightarrow \ln D = \ln k + \ln w + 3\ln l - 4\ln d \;\Rightarrow \frac{\Delta D}{D} = \frac{\Delta w}{w} + 3\frac{\Delta l}{l} - 4\frac{\Delta d}{d}$$

$$\Rightarrow 100\frac{\Delta D}{D} = 100\frac{\Delta w}{w} + 3\times 100\frac{\Delta l}{l} - 4\times 100\frac{\Delta d}{d}$$

$$= 3 + 3\times 2 - 4\times 1 = 5\%$$

Application activity 4.1

1. Find the differential of:

   a) $f(x) = x^2 - 3x$ b) $f(x) = \frac{2-x}{2+x}$

   c) $f(x) = \frac{3}{4}\sqrt{2-x}$

2. Find the percentage error in the area of a rectangle when an error of $+1$ per cent is made in measuring its length and breadth.

3. The period T of a simple pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$.

   Find the maximum error in T due to possible errors up to 1% in l and 2.5% in g.

4. The diameter and altitude of a can in the shape of a right circular cylinder are measured as 40 cm and 64 cm respectively. The possible error in each measurement is $\pm 5\%$. Find approximately the maximum possible error in the computed value for the volume and the lateral surface. Hence obtain the corresponding percentage error in each case.

4.2. Indefinite integrals

4.2.1. Definition

Activity 4.2

For each of the following functions, find function $g(x)$ such that $g'(x) = f(x)$.

a) $f(x) = 3x^2$

b) $f(x) = x$

c) $f(x) = \frac{1}{\sqrt{x}}$

d) $f(x) = -\frac{1}{x^2}$

An **integral** or an **anti-derivative** of function $f(x)$ is the function $F(x)$ whose derivative is equal to $f(x)$. Thus, we say $F(x)$ is an anti-derivative of $f(x)$ and write

$F'(x) = f(x)$.

The process of solving for anti-derivatives is called **anti-differentiation** (or **integration**) which is the opposite operation of differentiation (process of finding derivatives).

Example 4.4

The function $F(x) = \ln x$ is the primitive of $f(x) = \frac{1}{x}$ since $(\ln x)' = \frac{1}{x}$.

Also, $F(x) = \ln x + 5$ is the primitive of $f(x) = \frac{1}{x}$ since

$(\ln x + 5)' = \frac{1}{x} + 0 = \frac{1}{x}$.

Recall that the derivative of a constant is zero.

Also, $F(x) = \ln x - 20$ is the primitive of $f(x) = \frac{1}{x}$ since

$(\ln x - 20)' = \frac{1}{x} - 0 = \frac{1}{x}$.

Thus, every correct integral of $f(x)$ has the form $F(x) + c$ where c is an arbitrary constant and $F'(x) = f(x)$

Example 4.5

The primitive function of $f(x) = \cos x$ is $F(x) = \sin x + c$ since $F'(x) = (\sin x + c)' = \cos x$.

Notation

The anti-derivative of $f(x)$ called the **indefinite integral** of $f(x)$ is denoted by

$\int f(x)dx$ so that $\int f(x)dx = F(x) + c$

where

$\int$ is the **integral sign**, $f(x)dx$ is called the **integrand**, x is the **variable of integration**, $F'(x) = f(x)$, c is the **constant of integration** as its value is not known, unless we have further information.

Such integrals where we add an arbitrary constant to every correct result are called **indefinite integrals**.

Example 4.6

Find $\int \frac{dx}{1+x^2}$

Solution

$\int \frac{dx}{1+x^2} = \arctan x + c$. Indeed $(\arctan x + c)' = \frac{1}{1+x^2}$

Example 4.7

Find $\int xdx$

Solution

$\int xdx = \frac{x^2}{2} + c$. Indeed $\left(\frac{x^2}{2} + c\right)' = x$

Application activity 4.2

Find each of the following integrals:

1. $\int (4x-5)\,dx$
2. $\int (6x^2+4x+3)\,dx$
3. $\int (x^3+x^2+x)\,dx$
4. $\int (3x-4)^2\,dx$
5. $\int 5\,dx$
6. $\int (2x-3)(3-4x)\,dx$
7. $\int (x^2+1)(2x^2-5)\,dx$
8. $\int (2x^2-1)^2\,dx$
9. $\int (x-2)^3\,dx$

4.2.2. Properties of integrals

Activity 4.3

1. For $f(x)=\cos x$, find the derivative of $\int f(x)\,dx$. Give your observation.
2. For $f(x)=\sin x$, find the integral of the differential of $f(x)$. Give your observation.
3. For $f(x)=3x$, find $\int 3x\,dx$. Compare your result with $3\int x\,dx$.
4. For $f(x)=x^3+3x-1$ and $g(x)=x^2+2x+2$, find $\int f(x)\,dx+\int g(x)\,dx$ and $\int [f(x)+g(x)]\,dx$. Give your observation.
5. Find $\frac{d}{dx}\cos(2x+3)$. Deduce $\int [-\sin(2x+3)]\,dx$.
6. Hence, write down the formula that could be used to find $\int f(ax+b)\,dx,\ a,b\in\mathbb{R},\ a\neq 0$ when $F(x)$ is the integral of $f(x)$.

From activity 4.3, we get the following important properties:

1. The derivative of the indefinite integral is equal to the function to be integrated.

$$\frac{d}{dx}\int f(x)\,dx = f(x)$$

2. The integral of differential of a function is equal to the sum of that function and an arbitrary constant.

$$\int df(x) = f(x) + c$$

3. Each constant function may be pulled out of integral sign.

$$\int kf(x)\,dx = k\int f(x)\,dx$$

4. The indefinite integral of the algebraic sum of two functions is equal to the algebraic sum of the indefinite integrals of those functions.

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

5. If $F(x)$ is an indefinite integral of $f(x)$, then, the integral

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + c \qquad a,b,c \in \mathbb{R}, a \neq 0$$

Example 4.8

Find $\int \cos 3x dx$

Solution

$$\int \cos 3x dx = \frac{1}{3}\sin 3x + c$$

Example 4.9

Find $\int e^{2x} dx$

Solution

$$\int e^{2x} dx = \frac{1}{2}e^{2x} + c$$

Example 4.10

Find $\frac{d}{dx}\int \sqrt{x^2+3}\,dx$

Solution

$$\frac{d}{dx}\int \sqrt{x^2+3}dx = \sqrt{x^2+3}$$

Example 4.11

Find $\int d\sqrt{x^2+3}$

Solution

$$\int d\sqrt{x^2+3} = \sqrt{x^2+3} + c$$

Application activity 4.3

If $\int f(x)dx = x^2 + 2x + c$, $\int g(x)dx = x^3 - 3x^2 - 4x + k$ find;

1. $4\int f(x)dx$
2. $\frac{2}{5}\int [g(x)-6]dx$
3. $\int [f(x)+3g(x)]dx$
4. $\frac{d}{dx}\int [2f(x)-3g(x)]dx$

4.3. Techniques of integration

4.3.1. Integration by substitution#

Activity 4.4

Consider the integral $\int e^{5x+2}\,dx$. By letting $u = 5x+2$ and differentiating u with respect to x, find this integral.

Integration by substitution is based on rule for differentiating composite functions. The formula for integration by substitution is

$$\int f(x)dx = \int f(x(t))x'(t)dt$$

Basic integrals of exponential functions

From the knowledge of differential calculus, we can give the following results:

1. $\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$
2. $\int e^x dx = e^x + c$
3. $\int a^x dx = \frac{a^x}{\ln a} + c$

Example 4.12

Find $\int e^{2x} dx$

Solution

Let $t=2x$, so $dt=2dx \Rightarrow dx=\frac{1}{2}dt$

We have, $\int e^{2x}dx=\frac{1}{2}\int e^{t}dt=\frac{1}{2}e^{2x}+c$

Example 4.13

Find $\int (2x+1)^4\,dx$

Solution

Let $t=2x+1$, so $dt=2dx \Rightarrow dx=\frac{1}{2}dt$

We have,

$$\int (2x+1)^4\,dx=\frac{1}{2}\int t^4 dt=\frac{1}{2}\left(\frac{t^5}{5}\right)+c=\frac{1}{10}(2x+1)^5+c$$

Example 4.14

Find $\int \frac{x^3}{(x-1)^2}dx$

Solution

Let $t=x-1 \Rightarrow x=t+1$, so $dt=dx \Rightarrow dx=dt$

We have,

$$\int \frac{x^3}{(x-1)^2}dx=\int \frac{(t+1)^3}{t^2}dt \quad =\int \frac{t^3+3t^2+3t+1}{t^2}dt$$

$$=\int\left(t+3+\frac{3}{t}+\frac{1}{t^2}\right)dt \quad =\int tdt+\int 3dt+\int \frac{3}{t}dt+\int \frac{1}{t^2}dt$$

$$=\frac{t^2}{2}+3t+3\ln|t|-t^{-1}+c \quad =\frac{(x-1)^2}{2}+3(x-1)+3\ln|x-1|-\frac{1}{x-1}+c$$

Application activity 4.4

1. Evaluate;

a) $\int\left(e^x - x^e\right)dx$ b) $\int x^2 e^{x^3}\,dx$ c) $\int\left(e^x+1\right)^2 dx$

d) $\int \frac{e^{\frac{1}{x^2}}}{x^3}dx$ e) $\int e^x \cos\left(e^x\right)dx$ f) $\int e^{3\cos 2x}\sin 2x\,dx$

g) $\int \frac{\cos\left(\ln x\right)}{x}dx$ h) $\int\left(4x^3-12\right)^2 x^2 dx$

2. A particle moves in a straight line such that its velocity at time t seconds is given by $v = \frac{100t}{\left(t^2+1\right)^3}\,m\,s^{-1}$.

 Find the distance travelled by the particle in the first two seconds of motion.

4.2.2. Integration of rational functions

A function $f\left(x\right) = \frac{g\left(x\right)}{h\left(x\right)}$, where $g\left(x\right)$ *and* $h\left(x\right) \neq 0$ are polynomials, is called a rational function. When integrating a rational function, we need to check if there is a relationship between the numerator and the derivative of the denominator.

Basic integrals of rational functions

From the knowledge of differential calculus, we can give the following table of results:

1. $\int \frac{1}{x}dx = \ln\left|x\right| + c$
2. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + c$
3. $-\int \frac{dx}{a^2+x^2} = \frac{1}{a}\operatorname{arc\,cot}\frac{x}{a} + c$

Numerator can be expressed in terms of derivative of the denominator

Activity 4.5

From derivative of reciprocal functions and logarithmic derivative, find;

1. $\int \frac{x}{\left(1-x^2\right)^2}dx$

2. $\int \frac{2x-1}{3x^2-3x+1}dx$

Since $(\ln u)'=\frac{u'}{u}$ and $\left(\frac{1}{u}\right)'=-\frac{u'}{u^2}$, thus,

$\int \frac{u'}{u}dx=\int(\ln u)'dx=\ln u+c$ and $\int \frac{u'}{u^2}dx=-\frac{1}{u}+c$

In activity 4.5, the following basic integration formulae are most helpful:

$\int \frac{u'}{u}dx=\ln|u|+c$, $\int \frac{u'}{u^2}dx=-\frac{1}{u}+c$ and $\int \frac{u'}{u^2+1}dx=\arctan u+c$

Example 4.15

Find $\int\left(2-3x^2+\frac{1}{x}-\frac{4}{x^2+1}\right)dx$

Solution

$$\int\left(2-3x^2+\frac{1}{x}-\frac{4}{x^2+1}\right)dx=\int 2dx-\int 3x^2dx+\int\frac{1}{x}dx-\int\frac{4}{x^2+1}dx$$

$$=2\int dx-3\int x^2dx+\int\frac{1}{x}dx-4\int\frac{1}{x^2+1}$$

$$=2x-x^2+\ln|x|-4\arctan x+c$$

Example 4.16

Find $\int\frac{x+1}{x^2+2x+3}dx$

Solution

Here, $\left(x^2+2x+3\right)'=2x+2=2\left(x+1\right)$

We can write

$$\int\frac{x+1}{x^2+2x+3}dx=\int\frac{\frac{1}{2}\left(2x+2\right)}{x^2+2x+3}dx=\frac{1}{2}\int\frac{\left(x^2+2x+3\right)'}{x^2+2x+3}dx$$

$$=\frac{1}{2}\ln\left|x^2+2x+3\right|+c=\ln\sqrt{x^2+2x+3}+c$$

Example 4.17

Find $\int\frac{1}{x^2+2x+1}dx$

Solution

Here, $x^2+2x+1=\left(x+1\right)^2$. But $\left(x+1\right)'=1$

We can write

$$\int\frac{1}{x^2+2x+1}dx=\int\frac{1}{\left(x+1\right)^2}dx=\int\frac{(x+1)'}{\left(x+1\right)^2}dx=-\frac{1}{x+1}+c$$

Application activity 4.5

Evaluate:

1. $\int\frac{\left(x+1\right)}{\left(x^2+2x+3\right)^2}dx$

2. $\int\frac{x}{\left(1-x^2\right)^2}dx$

3. $\int\frac{x^2}{\left(2x^3+5\right)^2}dx$

4. $\int\frac{x+1}{\left(x^2+2x+5\right)^3}dx$

Numerator being not expressible in terms of derivative of the denominator

An improper rational fraction (where the degree of the numerator is greater than or equal to the degree of the denominator) can be expressed as a sum of simpler fractions (partial fractions) whose denominators are of the form $\left(ax+b\right)^n$ and $\left(ax^2+bx+c\right)^n$, n being a positive integer.

Two cases arise:

Case 1: Degree of the numerator is greater than or equal to the degree of the denominator

Activity 4.6

Recall that if quotient of the division $\frac{f(x)}{g(x)}$ is $q(x)$ and remainder is $r(x)$, then, $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$.

Use long division to write the equivalent expression for

1. $\frac{2x+4}{5x-3}$
2. $\frac{x^2-3x+2}{x^2+1}$
3. $\frac{x^2+1}{x-1}$
4. $\frac{x^3+2x-4}{x^2+2}$

Hence, deduce their anti-derivatives.

To integrate a rational function where the degree of numerator is greater than or equal to the degree of denominator, we proceed by long division.

Example 4.18

Find $\int \frac{x^2}{x+1} dx$

Solution

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2 \\ & \underline{-x^2-x} \\ & \quad -x \\ & \quad \underline{x+1} \\ & \qquad 1 \end{array}$$

$$\Rightarrow \frac{x^2}{x+1} = x-1+\frac{1}{x+1}$$

Example 4.19

Find $\int \frac{x+1}{x-1} dx$

Solution

$$\begin{array}{r|l} & 1 \\ \hline x-1 & x+1 \\ & \underline{-x+1} \\ & \quad 2 \end{array}$$

$$\Rightarrow \frac{x+1}{x-1} = 1+\frac{2}{x-1}$$

Then,

$$\int \frac{x^2}{x+1}dx = \int\left(x-1+\frac{1}{x+1}\right)dx$$

$$= \int xdx - \int dx + \int \frac{1}{x+1}dx$$

$$= \frac{x^2}{2} - x + \ln|x+1| + c$$

Then,

$$\int \frac{x+1}{x-1}dx = \int\left(1+\frac{2}{x-1}\right)dx$$

$$= \int dx + \int \frac{2}{x-1}dx$$

$$= x + 2\ln|x-1| + c$$

Application activity 4.6

Evaluate the following integrals:

1. $\int \frac{x^3-2}{x^2+1}dx$
2. $\int \frac{x^2-2}{x^2+x-2}dx$
3. $\int \frac{x^2+1}{6x-9x^2}dx$
4. $\int \frac{x^5}{x^3-a^3}dx$
5. $\int \frac{x^3+1}{x^2+7x+12}dx$

Case 2: Degree of the numerator is less than degree of the denominator

In this case, we reduce the fraction in simple fractions. The first step is to factorise the denominator.

In this case, the following situations my arise:

A. The denominator is factorised into linear factors

Activity 4.7

Factorise completely the denominator and then decompose the given fraction into partial fractions:

1. $\frac{x-2}{x^2+2x}$
2. $\frac{x}{x^2+3x+2}$
3. $\frac{2}{x^2-1}$
4. $\frac{2x-3}{x^2-x-2}$

Hence or otherwise, find their anti-derivatives.

To each factor $ax+b$ occurring once in the denominator of a proper rational fraction, there is corresponding single partial fraction of the form $\frac{A}{ax+b}$; where A is a constant to be found. But to each factor $ax+b$ occurring n times in the denominator of a proper rational fraction, there corresponds a sum of n partial fractions of the form

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\ldots+\frac{A_n}{(ax+b)^n};$$

where A_n are constants to be found.

Example 4.20

Find $\int \frac{x+3}{x^2-5x+4}dx$

Solution

We need to factorise x^2-5x+4. That is, $x^2-5x+4=(x-4)(x-1)$

Then,

$$\int \frac{x+3}{x^2-5x+4}dx=\int \frac{x+3}{(x-4)(x-1)}dx$$

Let $\frac{x+3}{(x-4)(x-1)}=\frac{A}{x-4}+\frac{B}{x-1}$

$$\Leftrightarrow \frac{x+3}{(x-4)(x-1)}=\frac{A(x-1)+B(x-4)}{(x-4)(x-1)}$$

$$\Leftrightarrow x+3=A(x-1)+B(x-4) \quad \Leftrightarrow x+3=Ax-A+Bx-4B$$

$$\Leftrightarrow x+3=(A+B)x-A-4B$$

$$\begin{cases} A+B=1 \\ \underline{-A-4B=3} \end{cases}$$

$$-3B=4 \Rightarrow B=-\frac{4}{3}$$

$$A = 1 - B = 1 + \frac{4}{3} = \frac{7}{3}$$

And

$$\frac{x+3}{(x-4)(x-1)} = \frac{\frac{7}{3}}{x-4} + \frac{\frac{-4}{3}}{x-1}$$

Now,

$$\int \frac{x+3}{x^2-5x+4}dx = \int \left(\frac{\frac{7}{3}}{x-4} + \frac{\frac{-4}{3}}{x-1} \right) dx$$

$$= \frac{7}{3}\int \frac{1}{x-4}dx - \frac{4}{3}\int \frac{1}{x-1}dx \quad = \frac{7}{3}\ln|x-4| - \frac{4}{3}\ln|x-1| + c$$

$$= \frac{1}{3}\ln\left|(x-4)^7\right| - \frac{1}{3}\ln\left|(x-1)^4\right| + c \quad = \ln\left|\sqrt[3]{(x-4)^7}\right| - \ln\left|\sqrt[3]{(x-1)^4}\right| + c$$

$$= \ln\left|\sqrt[3]{\frac{(x-4)^7}{(x-1)^4}}\right| + c$$

Example 4.21

Find $\int \frac{dx}{x^2-4}$

Solution

Since $x^2 - 4 = (x-2)(x+2)$,

$$\frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

$$\Leftrightarrow 1 = A(x+2) + B(x-2)$$

Take $x = -2$,

$$1 = A(-2+2) + B(-2-2)$$

$$\Leftrightarrow 1 = -4B \Rightarrow B = \frac{-1}{4}$$

Take $x=2$,

$$1=A(2+2)+B(2-2)$$

$$\Leftrightarrow 1=4A \Rightarrow A=\frac{1}{4}$$

Then, $\dfrac{1}{x^2-4}=\dfrac{\frac{1}{4}}{x-2}+\dfrac{\frac{-1}{4}}{x+2}$

$$\Rightarrow \int \frac{dx}{x^2-4}=\int \frac{\frac{1}{4}}{x-2}dx+\int \frac{\frac{-1}{4}}{x+2}dx$$

$$=\frac{1}{4}\ln|x-2|-\frac{1}{4}\ln|x+2|+c=\ln\sqrt[4]{x-2}-\ln\sqrt[4]{x+2}+c \qquad =\ln\sqrt[4]{\frac{x-2}{x+2}}+c$$

Alternative method

$$\int \frac{dx}{x^2-4}$$

$$\int \frac{dx}{x^2-4}=\int \frac{dx}{x^2-2^2}$$

But, $\displaystyle\int \frac{dx}{x^2-a^2}=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+c$

Then,

$$\int \frac{dx}{x^2-2^2}=\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+c=\ln\sqrt[4]{\frac{x-2}{x+2}}+c$$

Example 4.22

Find $\displaystyle\int \frac{2x+2}{x^2+2x+1}dx$

Solution

Since $x^2+2x+1=(x+1)^2$,

$$\frac{2x+2}{(x+1)^2}=\frac{A}{x+1}+\frac{B}{(x+1)^2} \Leftrightarrow \frac{2x+2}{(x+1)^2}=\frac{A(x+1)+B}{(x+1)^2} \Leftrightarrow 2x+2=A(x+1)+B$$

$\Leftrightarrow 2x+2 = A(x+1)+B \Leftrightarrow 2x+2 \equiv Ax+A+B$

$$\begin{cases} A=2 \\ A+B=2 \Rightarrow B=0 \end{cases}$$

Then, $\dfrac{2x+2}{x^2+2x+1} = \dfrac{2}{x+1}$

$$\int \frac{2x+2}{x^2+2x+1}dx = \int \frac{2}{x+1}dx = 2\ln|x+1|+c = \ln(x+1)^2+c = \ln(x^2+2x+1)+c$$

Alternative method

We see that $(x^2+2x+1)' = 2x+2$

Then,

$$\int \frac{2x+2}{x^2+2x+1}dx = \int \frac{(x^2+2x+1)'}{x^2+2x+1}dx = \ln|x^2+2x+1|+c$$

Example 4.23

Find $\displaystyle\int \frac{8x^3-16x+1}{x^4-8x^2+16}dx$

Solution

$x^4-8x^2+16 = (x-2)^2(x+2)^2$

$$\frac{8x^3-16x+1}{(x-2)^2(x+2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+2} + \frac{D}{(x+2)^2}$$

$\Leftrightarrow 8x^3-16x+1 = A(x-2)(x+2)^2 + B(x+2)^2 + C(x+2)(x-2)^2 + D(x-2)^2$

$\Leftrightarrow 8x^3-16x+1 = A(x^3+2x^2-4x-8) + B(x^2+4x+4) + C(x^3-2x^2-4x+8) + D(x^2-4x-4)$

$\Leftrightarrow 8x^3-16x+1 = (A+C)x^3 + (2A+B-2C+D)x^2 + (-4A+4B-4C-4D)x + (-8A+4B+8C-4D)$

$$\begin{cases} A+C=8 \\ 2A+B-2C+D=0 \\ -4A+4B-4C-4D=-16 \\ -8A+4B+8C+4D=1 \end{cases}$$

From which we get;

$$A=\frac{127}{32},\ B=\frac{33}{16},\ C=\frac{129}{32},\ D=-\frac{31}{16}$$

Thus,

$$\int\frac{8x^3-16x+1}{x^4-8x^2+16}dx=\int\left(\frac{\frac{127}{32}}{x-2}+\frac{\frac{33}{16}}{(x-2)^2}+\frac{\frac{129}{32}}{x+2}+\frac{-\frac{31}{16}}{(x+2)^2}\right)dx$$

$$=\frac{127}{32}\int\frac{dx}{x-2}+\frac{33}{16}\int\frac{dx}{(x-2)^2}+\frac{129}{32}\int\frac{dx}{x+2}-\frac{31}{16}\int\frac{dx}{(x+2)^2}$$

$$=\frac{127}{32}\ln|x-2|-\frac{33}{16}\left(\frac{1}{x-2}\right)+\frac{129}{32}\ln|x+2|+\frac{31}{16}\left(\frac{1}{x+2}\right)+c$$

$$=\frac{127}{32}\ln|x-2|+\frac{129}{32}\ln|x+2|+\frac{31}{16}\left(\frac{1}{x+2}\right)-\frac{33}{16}\left(\frac{1}{x-2}\right)+c$$

Application activity 4.7

Find:

1. $\int\frac{2}{x^2-1}dx$
2. $\int\frac{x}{x^2+3x+2}dx$
3. $\int\frac{x-3}{-x^2+2x}dx$
4. $\int\frac{x}{x^2+2x+1}dx$
5. $\int\frac{3x}{x^2-4x+4}dx$
6. $\int\frac{8x^2-19x}{x^3-3x^2+4}dx$

B. The denominator is a quadratic factor

Activity 4.8

Recall (in senior four) that, for $a\neq 0, b, c\in\mathbb{R}$,

$$ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]$$

Using this relation, transform denominator of each of the following integrals and then integrate

1. $\int \frac{dx}{x^2+3x+2}$ 2. $\int \frac{dx}{x^2-4x+4}$ 3. $\int \frac{dx}{x^2-6x+18}$

Consider the following :

- In each case, put $u = x + \frac{b}{2a}$

From activity 4.8, by taking the integral of the form $\int \frac{dx}{ax^2+bx+c}$,

- If $b^2-4ac=0$, then

$$\int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2}$$ and we let $u = x + \frac{b}{2a}$.

- If $b^2-4ac>0$, then

$$\int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2+\frac{b^2-4ac}{4a^2}}.$$

We let $u = x + \frac{b}{2a}$, $k^2 = \frac{b^2-4ac}{4a^2}$ and use the standard integral $\int \frac{dx}{x^2-k^2} = \frac{1}{2k}\ln\left|\frac{x-k}{x+k}\right| + d$

- If $b^2-4ac<0$, then,

$$\int \frac{dx}{ax^2+bx+c} = \frac{1}{a}\int \frac{dx}{\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}}.$$

We let $u = x + \frac{b}{2a}$, $-k^2 = \frac{b^2-4ac}{4a^2}$ and use the standard integral $\int \frac{dx}{x^2+k^2} = \frac{1}{k}\arctan\frac{x}{k} + d$.

Notice

- If there are other factors, to each irreducible quadratic factor ax^2+bx+c occurring once in the denominator of a proper fraction, there corresponds a single partial fraction of the form $\dfrac{Ax+B}{ax^2+bx+c}$, where A and B are constants to be found.
- To each irreducible quadratic factor ax^2+bx+c occurring n times in the denominator of a proper fraction, there correspnds a sum of n partial fractions of the form
$$\frac{A_1x+B_1}{x^2+bx+c}+\frac{A_2x+B_2}{\left(ax^2+bx+c\right)^2}+\ldots+\frac{A_nx+B_n}{\left(ax^2+bx+c\right)}$$
where A_n and B_n are constants to be found.

Example 4.24

Find $\displaystyle\int\frac{dx}{x^2-x+1}$

Solution

$a=1,\ b=-1,\ c=1$

$\Delta=(-1)^2-4(1)(1)=-3<0$

$x+\dfrac{b}{2a}=x-\dfrac{1}{2}$

Let $k^2=-\dfrac{\Delta}{4a^2}=-\dfrac{-3}{4}=\dfrac{3}{4}\Rightarrow k=\dfrac{\sqrt{3}}{2}$

$$\int\frac{dx}{x^2-x+1}=\frac{1}{\frac{\sqrt{3}}{2}}\arctan\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}+c=\frac{2}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}+c$$
$$=\frac{2\sqrt{3}}{3}\arctan\frac{\sqrt{3}(2x-1)}{3}+c$$

Example 4.25

Find $\displaystyle\int\frac{x^3}{x^4+2x^2+1}dx$

Solution

$x^4+2x^2+1=\left(x^2+1\right)^2$

$$\frac{x^3}{\left(x^2+1\right)^2}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{\left(x^2+1\right)^2}$$

$\Leftrightarrow x^3=\left(Ax+B\right)\left(x^2+1\right)+Cx+D$

$\Leftrightarrow x^3=Ax^3+Ax+Bx^2+B+Cx+D$

$\Leftrightarrow x^3=Ax^3+Bx^2+\left(A+C\right)x+B+D$

$$\begin{cases}A=1\\B=0\\A+C=0\\B+D=0\end{cases}\Rightarrow\begin{cases}A=1\\B=0\\C=-1\\D=0\end{cases}$$

Then,

$$\int\frac{x^3}{x^4+2x^2+1}dx=\int\frac{x}{x^2+1}dx+\int\frac{-x}{\left(x^2+1\right)^2}dx$$

$$=\int\frac{\frac{1}{2}(2x)}{x^2+1}dx-\int\frac{\frac{1}{2}(2x)}{\left(x^2+1\right)^2}dx=\frac{1}{2}\int\frac{\left(x^2+1\right)'}{x^2+1}dx-\frac{1}{2}\int\frac{\left(x^2+1\right)'}{\left(x^2+1\right)^2}dx$$

$$=\frac{1}{2}\ln\left|x^2+1\right|+\frac{1}{2}\frac{1}{x^2+1}+c=\ln\sqrt{x^2+1}+\frac{1}{2\left(x^2+1\right)}+c$$

Example 4.26

Find $\int\frac{x^2+2}{x^3-1}dx$

Solution

$x^3-1=(x-1)\left(x^2+x+1\right)$

$$\frac{x^2+2}{(x-1)\left(x^2+x+1\right)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$$

$\Leftrightarrow x^2+2=A\left(x^2+x+1\right)+\left(Bx+C\right)\left(x-1\right)$

Take $x=1$,

$3=3A \Rightarrow A=1$

Take $x=0$,

$2=A(1)+C(-1)$

$\Leftrightarrow 2=A-C \Leftrightarrow C=A-2 \Rightarrow C=-1$

Take $x=2$,

$6=A(7)+(2B+C)(1)$

$\Leftrightarrow 6=7A+2B+C \Leftrightarrow 2B=6-7A+C \Rightarrow B=0$

Then,

$$\int \frac{x^2+2}{x^3-1}dx=\int \frac{dx}{x-1}+\int \frac{dx}{x^2+x+1}=\ln|x-1|-\int \frac{dx}{x^2+x+1}$$

We need to calculate $\int \frac{dx}{x^2+x+1}$

$a=b=c=1$

$\Delta=1^2-4=-3$

$x+\frac{b}{2a}=x+\frac{1}{2}$

Let $k^2=-\frac{\Delta}{4a^2}=\frac{3}{4} \Rightarrow k=\frac{\sqrt{3}}{2}$

$$\int \frac{dx}{x^2+x+1}=\frac{1}{\frac{\sqrt{3}}{2}}\arctan\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}+c=\frac{2\sqrt{3}}{3}\arctan\frac{\sqrt{3}(2x+1)}{3}+c$$

Hence,

$$\int \frac{x^2+2}{x^3-1}dx=\ln|x-1|-\frac{2\sqrt{3}}{3}\arctan\frac{\sqrt{3}(2x+1)}{3}+c$$

Example 4.27

Find $\int \frac{dx}{6x^2-5x+1}$

Solution

$a = 6,\ b = -5,\ c = 1$

$b^2 - 4ac = 25 - 24 = 1 > 0$

Let $u = x + \dfrac{b}{2a} = x - \dfrac{5}{12} \Rightarrow du = dx$, $k^2 = \dfrac{b^2 - 4ac}{4a^2} = \dfrac{1}{144}$

$$\int \frac{dx}{6x^2 - 5x + 1} = \frac{1}{6}\int \frac{du}{u^2 - \frac{1}{44}} \quad = \frac{1}{6}\int \frac{du}{u^2 - \left(\frac{1}{12}\right)^2}$$

$$= \frac{1}{6} \times \frac{1}{2 \times \frac{1}{12}} \ln\left|\frac{u - \frac{1}{12}}{u + \frac{1}{12}}\right| + c \quad = \ln\left|\frac{12u - 1}{12u + 1}\right| + c$$

$$= \ln\left|\frac{12\left(x - \frac{5}{12}\right) - 1}{12\left(x - \frac{5}{12}\right) + 1}\right| + c \quad = \ln\left|\frac{12x - 5 - 1}{12x - 5 + 1}\right| + c$$

$$= \ln\left|\frac{6(2x-1)}{4(3x-1)}\right| + c \quad = \ln\left|\frac{2x-1}{3x-1} \times \frac{6}{4}\right| + c$$

$$= \ln\left|\frac{2x-1}{3x-1}\right| + \ln\frac{6}{4} + c$$

$$= \ln\left|\frac{2x-1}{3x-1}\right| + d \qquad \left[\text{since } \ln\frac{6}{4} \text{ is another constant}\right]$$

Application activity 4.8

Find:

1. $\displaystyle\int \frac{1}{x^2 + x + 2}\,dx$

2. $\displaystyle\int \frac{x}{9x^2 + 6x + 2}\,dx$

3. $\displaystyle\int \frac{6x^2 - x + 5}{(x-2)(2x^2+1)}\,dx$

4. $\displaystyle\int \frac{x - 4}{(2x+1)(x^2+2)}\,dx$

4.3.3. Integration of trigonometric functions

Basic integrals of trigonometric functions

From the knowledge of differential calculus, we can give the following table of results:

1. $\int \sin x\,dx = -\cos x + c$
2. $\int \cos x\,dx = \sin x + c$
3. $\int \sec^2 x\,dx = \tan x + c$
4. $\int \csc^2 x\,dx = -\cot x + c$
5. $\int \tan x\,dx = -\ln|\cos x| + c$
6. $\int \cot x\,dx = \ln|\sin x| + c$
7. $\int \sec x\,dx = \ln|\sec x + \tan x| + c$
8. $\int \csc x\,dx = -\ln|\csc x + \cot x| + c$
9. $\int \sec x \tan x\,dx = \sec x + c$
10. $\int \csc x \cot x\,dx = -\csc x + c$

Integrals of the form $\int \sin mx \cos nxdx$ or $\int \cos mx \cos nxdx$ or $\int \sin mx \sin nxdx$; m and n are constants

Activity 4.9

For each of the following functions, transform the product into sum and hence find the integral $\int f(x)dx$.

1. $f(x) = \sin 2x \cos x$
2. $f(x) = \sin x \sin 5x$
3. $f(x) = \cos 2x \cos 3x$
4. $f(x) = \sin x \sin 3x \sin 4x$

To evaluate the integral of the form $\int \sin mx \cos nxdx$

or $\int \cos mx \cos nxdx$ or $\int \sin mx \sin nxdx$, we use the corresponding identities:

$$\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

$$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\cos A\cos B=\frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$$

Example 4.28

Find $\int \cos 3x \sin 5x dx$

Solution

$$\cos 3x\sin 5x=\frac{1}{2}\left[\sin 8x-\sin(-2x)\right]=\frac{1}{2}(\sin 8x+\sin 2x)$$

Then,

$$\int \cos 3x\sin 5x dx=\int \frac{1}{2}(\sin 8x+\sin 2x)\,dx$$

$$=\frac{1}{2}\int \sin 8x dx+\frac{1}{2}\int \sin 2x dx$$

$$=\frac{1}{2}\left(-\frac{1}{8}\cos 8x\right)+\left(-\frac{1}{2}\cos 2x\right)\;=-\frac{1}{16}\cos 8x-\frac{1}{4}\cos 2x+c$$

Example 4.29

Find $\int \sin x \sin 2x \sin 3x dx$

Solution

$$\sin x\sin 2x=\frac{1}{2}(\cos x-\cos 3x)$$

$$\sin x\sin 2x\sin 3x=\frac{1}{2}(\cos x-\cos 3x)\sin 3x$$

$$=\frac{1}{2}(\cos x\sin 3x-\cos 3x\sin 3x)$$

$$=\frac{1}{2}\left[\frac{1}{2}(\sin 4x-\sin(-2x))-\frac{1}{2}(\sin 6x-\sin 0)\right]$$

$$=\frac{1}{4}\sin 4x+\frac{1}{4}\sin 2x-\frac{1}{4}\sin 6x$$

Then,

$$\int \sin x \sin 2x \sin 3x dx = \frac{1}{4}\int \sin 4x dx + \frac{1}{4}\int \sin 2x dx - \frac{1}{4}\int \sin 6x dx$$

$$= -\frac{1}{16}\cos 4x - \frac{1}{8}\cos 2x + \frac{1}{24}\cos 6x + c$$

Example 4.30

Find $\int \cos^2 x \cos x dx$

Solution

$$\cos^2 x \cos x = \frac{1+\cos 2x}{2}\cos x = \frac{\cos x + \cos 2x \cos x}{2}$$

$$= \frac{\cos x + \frac{1}{2}(\cos 3x + \cos x)}{2} = \frac{1}{2}\cos x + \frac{1}{4}\cos 3x + \frac{1}{4}\cos x$$

$$= \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$$

Then,

$$\int \cos^2 x \cos x dx = \frac{3}{4}\int \cos x dx + \frac{1}{4}\int \cos 3x dx = \frac{3}{4}\sin x + \frac{1}{12}\sin 3x + c$$

Application activity 4.9

Find:

1. $\int \sin 3x \cos 2x \, dx$
2. $\int \sin 2x \cos 3x \, dx$
3. $\int \sin 3x \sin 3x \, dx$
4. $\int \sin x \cos x \, dx$
5. $\int \cos 3x \cos 3x \, dx$
6. $\int \cos x \cos 7x \, dx$

Integrals of the form $\int \sin^m x \cos^n x dx$ $(m, n \in \mathbb{Z}^+)$

Activity 4.10

1. By letting $u = \cos x$, find $\int \sin x \cos^2 x dx$
2. Use the identities $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and

   $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ to find $\int \sin^2 x \cos^2 x dx$

If m or n is odd, save one cosine factor (or one sine factor) and use the relation $\cos^2 x = 1 - \sin^2 x$ (or $\sin^2 x = 1 - \cos^2 x$). Then let $u = \sin x \Rightarrow du = \cos x dx$ (or let $u = \cos x \Rightarrow du = -\sin x dx$).

If m and n are both even, we use the identities: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$. It is sometimes helpful to use the identity

$\sin x \cos x = \frac{1}{2} \sin 2x$.

Example 4.31

Find $\int \sin^3 x \cos^2 x dx$

Solution

$$\int \sin^3 x \cos^2 x dx = \int \sin^2 x \cos^2 x \sin x dx$$
$$= \int \left(1 - \cos^2 x\right) \cos^2 x \sin x dx$$

Let

$u = \cos x \Rightarrow du = -\sin x dx$

$\Rightarrow \sin x dx = -du$

Then,

$$\int \sin^3 x \cos^2 x dx = \int \left(1 - u^2\right) u^2 (-du)$$
$$= -\int \left(u^2 - u^4\right) du = -\int u^2 du + \int u^5 du$$
$$= -\frac{u^3}{3} + \frac{u^5}{5} + c = -\frac{\cos^3 x}{3} + \frac{\cos^5}{5} + c$$

Example 4.32

Find $\int \cos^5 x dx$

Solution

$$\int \cos^5 x dx = \int \cos^4 x \cos x dx$$

$$= \int \left(1-\sin^2 x\right)^2 \cos x dx = \int \left(1-2\sin^2 x+\sin^4 x\right)\cos x dx$$

$$= \int \cos x dx - 2\int \sin^2 x \cos x dx + \int \sin^4 x \cos x dx$$

Let $u = \sin x \Rightarrow du = \cos x dx$

Then,

$$\int \cos^5 x dx = \sin x - 2\int u^2 du + \int u^4 \cos x dx$$

$$= \sin x - 2\frac{u^3}{3} + \frac{u^5}{5} + c = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c$$

Example 4.33

Find $\int \sin^4 x \cos^2 x dx$

Solution

Both powers are even.

$$\sin^4 x \cos^2 x = \left(\frac{1}{2}(1-\cos 2x)\right)^2 \frac{1}{2}(1+\cos 2x)$$

$$= \frac{1}{4}\left(1-2\cos 2x+\cos^2 2x\right)\frac{1}{2}(1+\cos 2x)$$

$$= \frac{1}{8}\left(1-2\cos 2x+\cos^2 2x+\cos 2x-2\cos^2 2x+\cos^3 2x\right)$$

$$= \frac{1}{8}\left(1-\cos 2x-\cos^2 2x+\cos^3 2x\right)$$

$$= \frac{1}{8}\left(1-\cos 2x-\frac{1}{2}(1+\cos 4x)+\frac{1}{2}(1+\cos 4x)\cos 2x\right)$$

$$= \frac{1}{8}\left(1-\cos 2x-\frac{1}{2}-\frac{1}{2}\cos 4x+\frac{1}{2}\cos 2x+\frac{1}{2}\cdot\frac{1}{2}(\cos 2x+\cos 6x)\right)$$

$$=\frac{1}{8}\left(\frac{1}{2}-\frac{1}{2}\cos 2x-\frac{1}{2}\cos 4x+\frac{1}{4}\cos 2x+\frac{1}{4}\cos 6x\right)$$
$$=\frac{1}{8}\left(\frac{1}{2}-\frac{1}{4}\cos 2x-\frac{1}{2}\cos 4x+\frac{1}{4}\cos 6x\right)$$
$$=\frac{1}{32}\cos 6x-\frac{1}{16}\cos 4x-\frac{1}{32}\cos 2x+\frac{1}{16}$$

Then,

$$\int \sin^4 x\cos^2 x dx=\frac{1}{32}\int \cos 6x dx-\frac{1}{16}\int \cos 4x dx-\frac{1}{32}\int \cos 2x dx+\frac{1}{16}\int dx$$
$$=\frac{1}{192}\sin 6x-\frac{1}{64}\sin 4x-\frac{1}{64}\sin 2x+\frac{x}{16}+c$$

Alternative method

Linearise the expression $\sin^4 x\cos^2 x$:

$$\sin^4 x\cos^2 x=\left(\frac{e^{ix}-e^{-ix}}{2i}\right)^4\left(\frac{e^{ix}+e^{-ix}}{2}\right)^2$$
$$=\left(\frac{e^{4ix}-4e^{3ix}e^{-ix}+6e^{2ix}e^{-2ix}-4e^{ix}e^{-3ix}+e^{-4ix}}{16}\right)\left(\frac{e^{2ix}+2e^{ix}e^{-ix}+e^{-2ix}}{4}\right)$$
$$=\left(\frac{e^{4ix}-4e^{2ix}+6-4e^{-2ix}+e^{-4ix}}{16}\right)\left(\frac{e^{2ix}+2+e^{-2ix}}{4}\right)$$
$$=\frac{1}{64}\begin{pmatrix} e^{6ix}+2e^{4ix}+e^{2ix}-4e^{4ix}-8e^{2ix}-4e^{0}+6e^{2ix}+12 \\ +6e^{-2ix}-4e^{0}-8e^{-2ix}-4e^{-4ix}+e^{-2ix}+2e^{-4ix}+e^{-6ix}\end{pmatrix}$$
$$=\frac{1}{64}\left(e^{6ix}-2e^{4ix}-e^{2ix}-4+12-e^{-2ix}-4-2e^{-4ix}+e^{-6ix}\right)$$
$$=\frac{1}{64}\left(e^{6ix}+e^{-6ix}-2\left(e^{4ix}+e^{-4ix}\right)-\left(e^{2ix}+e^{-2ix}\right)+4\right)$$
$$=\frac{1}{64}\left(e^{6ix}+e^{-6ix}\right)-\frac{2}{64}\left(e^{4ix}+e^{-4ix}\right)-\frac{1}{64}\left(e^{2ix}+e^{-2ix}\right)+\frac{4}{64}$$
$$=\frac{1}{32}\left(\frac{e^{6ix}+e^{-6ix}}{2}\right)-\frac{1}{16}\left(\frac{e^{4ix}+e^{-4ix}}{2}\right)-\frac{1}{32}\left(\frac{e^{2ix}+e^{-2ix}}{2}\right)+\frac{1}{16}$$
$$=\frac{1}{32}\cos 6x-\frac{1}{16}\cos 4x-\frac{1}{32}\cos 2x+\frac{1}{16}$$

Then,

$$\int \sin^4 x\cos^2 xdx = \frac{1}{32}\int \cos 6xdx - \frac{1}{16}\int \cos 4xdx - \frac{1}{32}\int \cos 2xdx + \frac{1}{16}\int dx$$
$$= \frac{1}{192}\sin 6x - \frac{1}{64}\sin 4x - \frac{1}{64}\sin 2x + \frac{x}{16} + c$$

Application activity 4.10

Find:

1. $\int \cos^3 x \sin x\, dx$
2. $\int \sin^4 2x \cos 2x\, dx$
3. $\int \sin^3 x\, dx$
4. $\int \cos^3 4x\, dx$
5. $\int \sin^3 x \cos^3 x\, dx$
6. $\int \cos^3 2x \sin^5 2x\, dx$

Integrals of the form $\int \tan^m x\sec^n xdx$ $(m,n \in \mathbb{Z}^+)$

Activity 4.11

1. By letting $u = \tan x$, find $\int \tan^2 xdx$.
2. By using the identity $\tan^2 x = \sec^2 x - 1$ and letting $u = \sec x$, find $\int \tan^5 xdx$.

If the power of secant is even, save a factor of $\sec^2 x$ and use the identity $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors of secant in terms of $\tan x$ and then use the substitution $u = \tan x$.

If the power of tangent is odd, save a factor of $\sec x \tan x$ and use the identity $\tan^2 x = \sec^2 x - 1$ to express the remaining factors of tangent in terms of $\sec x$ and then use the substitution $u = \sec x$.

Example 4.34

Find $\int \tan^5 x\sec^7 xdx$

Solution

$$\int \tan^5 x\sec^7 xdx = \int \tan^4 x\sec^6 x\sec x\tan xdx$$

$$= \int \left(\sec^2 x - 1\right)^2 \sec^6 x \sec x \tan x dx$$

$$= \int \left(\sec^4 x - 2\sec^2 x + 1\right) \sec^6 x \sec x \tan x dx$$

$$= \int \left(\sec^{10} x - 2\sec^8 x + \sec^6 x\right) \sec x \tan x dx$$

Let $u = \sec x \Rightarrow du = \sec x \tan x dx$

Then,

$$\int \tan^5 x \sec^7 x dx = \int \left(u^{10} - 2u^8 + u^6\right) du$$

$$= \frac{u^{11}}{11} - 2\frac{u^9}{9} + \frac{u^7}{7} + c = \frac{\sec^{11} x}{11} - \frac{2\sec^9 x}{9} + \frac{\sec^7 x}{7} + c$$

Example 4.35

Find $\int \tan^4 x dx$

Solution

Let

$$u = \tan x \Rightarrow du = \sec^2 x dx = \left(1 + \tan^2 x\right) dx = \left(1 + u^2\right) du \Rightarrow dx = \frac{du}{1 + u^2}$$

Then,

$$\int \tan^4 x dx = \int u^4 \frac{du}{1 + u^2} = \int \frac{u^4}{1 + u^2} du$$

By long division; $\dfrac{u^4}{1 + u^2} = u^2 - 1 + \dfrac{1}{1 + u^2}$

$$\int \frac{u^4}{1 + u^2} du = \int u^2 du - \int du + \int \frac{1}{1 + u^2} du = \frac{u^3}{3} - u + \arctan u + c$$

Hence,

$$\int \tan^4 x dx = \frac{\tan^3 x}{3} - \tan x + \arctan\left(\tan x\right) + c = \frac{\tan^3 x}{3} - \tan x + x + c$$

Example 4.36

Find $\int \tan^3 x dx$

Solution

Let $u = \sec x$

$\Rightarrow du = \sec x \tan x dx$

$$dx = \frac{du}{\sec x \tan x} = \frac{du}{u \tan x}$$

$$\int \tan^3 x dx = \int \tan^2 x \tan x \, dx = \int \left(\sec^2 x - 1\right) \tan x \, dx$$

$$= \int \left(u^2 - 1\right) \tan x \frac{du}{u \tan x}$$

$$= \int \left(\frac{u^2}{u} - \frac{1}{u}\right) du \; = \int u \, du - \int \frac{du}{u} \; = \frac{1}{2} u^2 - \ln|u| + c$$

$$= \frac{1}{2} \sec^2 x - \ln|\sec x| + c$$

Application activity 4.11

Find;

1. $\int \sec^2 x \tan x \, dx$
2. $\int \sec x \tan^2 x \, dx$
3. $\int \sec^3 x \tan x \, dx$
4. $\int \sec^3 x \tan^3 x \, dx$
5. $\int \sec^2 x \tan^2 x \, dx$
6. $\int \sec^4 x \tan^2 x \, dx$

Integrals containing $\sin x, \cos x, \tan x$ **on denominator**

Activity 4.12

Recall (in senior 5) that:

$$\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}, \; \cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}, \; \tan x = \frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}}$$

By using these identities, find $\int \frac{dx}{\sin x + \cos x + 1}$

Hint: Using substitution; $u = \tan \frac{x}{2} \Rightarrow \arctan u = \frac{x}{2}$

From activity 4.12, for integral containing $\sin x, \cos x, \tan x$ on denominator, we use identities;

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}},\ \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}},\ \tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$$

and we let $u = \tan\frac{x}{2} \Rightarrow \arctan u = \frac{x}{2}$

Example 4.37

Find $\int \frac{dx}{\sin x}$

Solution

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \Rightarrow \frac{1}{\sin x} = \frac{1+\tan^2\frac{x}{2}}{2\tan\frac{x}{2}}$$

And $\int \frac{dx}{\sin x} = \int \frac{1+\tan^2\frac{x}{2}}{2\tan\frac{x}{2}} dx$

Let

$$u = \tan\frac{x}{2} \Rightarrow \arctan u = \frac{x}{2} \Rightarrow \frac{du}{1+u^2} = \frac{dx}{2} \Rightarrow \frac{2du}{1+u^2} = dx$$

$$\int \frac{1+\tan^2\frac{x}{2}}{2\tan\frac{x}{2}} dx = \int \frac{1+u^2}{2u}\frac{2du}{1+u^2} = \int \frac{du}{u} = \ln|u| + c$$

Hence,

$$\int \frac{dx}{\sin x} = \ln\left|\tan\frac{x}{2}\right| + c$$

Example 4.38

Find $\int \frac{dx}{\sin x+\cos x+2}$

Solution

$$\sin x+\cos x+2=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}+2=\frac{2\tan\frac{x}{2}+1-\tan^2\frac{x}{2}+2+2\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$=\frac{2\tan\frac{x}{2}+\tan^2\frac{x}{2}+3}{1+\tan^2\frac{x}{2}}$$

$$\int\frac{dx}{\sin x+\cos x+2}=\int\frac{1+\tan^2\frac{x}{2}}{2\tan\frac{x}{2}+\tan^2\frac{x}{2}+3}dx$$

Let

$$u=\tan\frac{x}{2}\Rightarrow \arctan u=\frac{x}{2}\Rightarrow\frac{du}{1+u^2}=\frac{dx}{2}\Rightarrow dx=\frac{2du}{1+u^2}$$

$$\int\frac{1+\tan^2\frac{x}{2}}{2\tan\frac{x}{2}+\tan^2\frac{x}{2}+3}dx=\int\frac{1+u^2}{2u+u^2+3}\times\frac{2du}{1+u^2}=2\int\frac{du}{u^2+2u+3}$$

$$=2\int\frac{du}{(u+1)^2+2}=2\frac{1}{\sqrt{2}}\arctan\frac{\sqrt{2}}{2}(u+1)+c=\sqrt{2}\arctan\frac{\sqrt{2}}{2}\left(\tan\frac{x}{2}+1\right)+c$$

Thus,

$$\int\frac{dx}{\sin x+\cos x+2}=\sqrt{2}\arctan\frac{\sqrt{2}}{2}\left(\tan\frac{x}{2}+1\right)+c$$

Application activity 4.12

Find the following integrals:

1. $\int \frac{dx}{2+\cos x}$
2. $\int \frac{dx}{3-2\sin x}$
3. $\int \frac{dx}{2+\sin 2x}$
4. $\int \frac{dx}{5-3\cos x}$
5. $\int \frac{dx}{3\cos x+4\sin x+6}$
6. $\int \frac{dx}{3\cos x-4\sin x+5}$

Integrals containing $\sin^2 x, \cos^2 x$ on denominator

Activity 4.13

Express $\cos x$ and $\sin x$ in terms of $\tan x$, hence integrate $\int \frac{1}{\cos^2 x} dx$

Hint: $\cos x = \frac{1}{\sec x}$ and $\sin x = \frac{\tan x}{\sec x}$

Letting $u = \tan x \Rightarrow x = \arctan u$

From activity 4.13, for integral containing $\sin^2 x$, $\cos^2 x$ on denominator, we use the identities $\cos x = \frac{1}{\sqrt{1+\tan^2 x}}$ and $\sin x = \frac{\tan x}{\sqrt{1+\tan^2 x}}$

and we let $u = \tan x \Rightarrow x = \arctan u$

Example 4.39

Find $\int \frac{dx}{\sin^2 x}$

Solution

$$\int \frac{dx}{\sin^2 x} = \int \frac{1+\tan^2 x}{\tan^2 x} dx$$

Let $u = \tan x \Rightarrow x = \arctan u$ *and* $dx = \frac{du}{1+u^2}$

$$\int \frac{dx}{\sin^2 x} = \int \frac{1+u^2}{u^2} \frac{du}{1+u^2} = \int \frac{du}{u^2} = \int \frac{u' du}{u^2}$$

$$= -\frac{1}{u} + c = -\frac{1}{\tan x} + c = -\cot x + c$$

Example 4.40

Find $\int \frac{dx}{\sin^4 x}$

Solution

$$\int \frac{dx}{\sin^4 x} = \int \frac{\left(1+\tan^2 x\right)^2 dx}{\tan^4 x}$$

Let $u = \tan x \Rightarrow x = \arctan u$ *and* $dx = \frac{du}{1+u^2}$

$$\int \frac{dx}{\sin^4 x} = \int \frac{\left(1+u^2\right)^2}{u^4} \frac{du}{1+u^2}$$

$$= \int \frac{\left(1+u^2\right) du}{u^4} = \int \frac{du}{u^4} + \int \frac{u^2}{u^4} du = \int u^{-4} du + \int u^{-2} du$$

$$= \frac{u^{-3}}{-3} + \frac{u^{-1}}{-1} + c = -\frac{1}{3u^3} - \frac{1}{u} + c = -\frac{1}{3\tan^3 x} - \frac{1}{\tan x} + c$$

$$= -\frac{\cot^3 x}{3} - \cot x + c$$

Application activity 4.13

Find the following indefinite integrals:

1) $\int \frac{1}{\cos^4 x} dx$ 2) $\int \frac{1}{\cos^6 x} dx$ 3) $\int \frac{1}{\sin^6 x} dx$ 4) $\int \frac{1}{\cos^8 x} dx$

Notice

Sometimes it is useful to use trigonometric identities and then use standard integrals.

Example 4.41

Find $\int\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2 dx$

Solution

$$\int\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2 dx=\int\left(\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}+\cos^2\frac{x}{2}\right)dx$$

$$=\int\left(\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}+\cos^2\frac{x}{2}\right)dx \;=\int\left(1+2\sin\frac{x}{2}\cos\frac{x}{2}\right)dx$$

$$=\int dx+\int 2\sin\frac{x}{2}\cos\frac{x}{2}dx \quad \left[\text{since } 2\sin\frac{x}{2}\cos\frac{x}{2}=\sin 2\frac{x}{2}=\sin x\right]$$

$$=\int dx+\int\sin x dx=x-\cos x+c$$

Example 4.42

Find $\int\tan^2 x\,dx$

Solution

$$\int\tan^2 x dx=\int\frac{\sin^2 x}{\cos^2 x}dx \;=\int\frac{1-\cos^2 x}{\cos^2 x}dx$$

$$=\int\frac{1}{\cos^2 x}dx-\int\frac{\cos^2 x}{\cos^2 x}=\tan x-x+c$$

4.3.4. Integration of irrational functions

Standard integrals of irrational functions

From the knowledge of differential calculus, we can write;

1. $\int\frac{dx}{\sqrt{a^2-x^2}}=\arcsin\frac{x}{a}+c$
2. $-\int\frac{dx}{\sqrt{a^2-x^2}}=\arccos\frac{x}{a}+c$
3. $\int\frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\text{arc}\sec\frac{x}{a}+c$
4. $-\int\frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\text{arc}\csc\frac{x}{a}+c$

Integrals containing $\sqrt[n]{ax+b},\ a \neq 0$

Activity 4.14

By letting $u^2 = 3x-1$, find $\int \sqrt{3x-1}\,dx$

When finding integral containing $\sqrt[n]{ax+b},\ a \neq 0$, we let $u^n = ax+b$.

Example 4.43

Find $\int \sqrt[3]{3x+1}\,dx$

Solution

Let $u^3 = 3x+1 \Rightarrow u = \sqrt[3]{3x+1}$

$\Rightarrow 3u^2du = 3dx \Leftrightarrow u^2du = dx$

$$\int \sqrt[3]{3x+1}\,dx = \int uu^2du$$

$$= \int u^3du \quad = \frac{u^4}{4} + c \quad = \frac{u^3u}{4} + c \quad = \frac{(3x+1)\sqrt[3]{3x+1}}{4} + c$$

Example 4.44

Find $\int \frac{x^2}{\sqrt{2x+1}}dx$

Solution

Let $u^2 = 2x+1 \Rightarrow u = \sqrt{2x+1}$

$\Rightarrow x = \frac{u^2-1}{2} \quad \Rightarrow dx = udu$

$$\int \frac{x^2}{\sqrt{2x+1}}dx = \int \frac{\left(\frac{u^2-1}{2}\right)^2}{u}udu \quad = \int \frac{u^4-2u^2+1}{4}du$$

$$= \int \frac{u^4}{4}du - \int \frac{u^2}{2}du + \int \frac{du}{4} \quad = \frac{u^5}{20} - \frac{u^3}{6} + \frac{u}{4} + c$$

$$=\frac{\left(\sqrt{2x+1}\right)^5}{20}-\frac{\left(\sqrt{2x+1}\right)^3}{6}+\frac{\sqrt{2x+1}}{4}+c$$

$$=\frac{\left(\sqrt{2x+1}\right)^4\sqrt{2x+1}}{20}-\frac{\left(\sqrt{2x+1}\right)^2\sqrt{2x+1}}{6}+\frac{\sqrt{2x+1}}{4}+c$$

$$=\frac{\left(4x^2+4x+1\right)\sqrt{2x+1}}{20}-\frac{\left(2x+1\right)\sqrt{2x+1}}{6}+\frac{\sqrt{2x+1}}{4}+c$$

$$=\sqrt{2x+1}\left(\frac{12x^2+12x+3-20x-10+15}{60}\right)+c$$

$$=\sqrt{2x+1}\left(\frac{12x^2-8x+8}{60}\right)+c \qquad =\sqrt{2x+1}\left(\frac{3x^2-2x+2}{15}\right)+c$$

Application activity 4.14

Find:

1. $\int\sqrt{6x+3}dx$
2. $\int\sqrt{(5x-2)^3}\,dx$
3. $\int\frac{1}{\sqrt[3]{8x+1}}dx$
4. $\int\frac{1}{\sqrt{(2-3x)^3}}dx$
5. $\int\sqrt{2x+5}dx$
6. $\int\sqrt[3]{3x-8}dx$
7. $\int\frac{4}{\sqrt{2x+3}}dx$
8. $\int\frac{2}{\sqrt{(1-3x)^3}}dx$

Integrals containing $\sqrt{ax^2+bx+c},\ a\neq 0$

Activity 4.15

Recall (in senior four) that, for $a\neq 0, b, c\in\mathbb{R}$,

$$ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right]$$

Using this relation, transform denominator of each of the following integrals and then integrate

1. $\int\frac{dx}{\sqrt{x^2-2x+1}}$
2. $\int\frac{dx}{\sqrt{x^2-5x+6}}$
3. $\int\frac{dx}{\sqrt{x^2-6x+18}}$

Recall (in senior four) that, for $a \neq 0, b, c \in \mathbb{R}$,

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right]$$

Using this relation, transform denominator of each of the following integrals and then integrate

1. $\int \frac{dx}{\sqrt{x^2 - 2x + 1}}$ 2. $\int \frac{dx}{\sqrt{x^2 - 5x + 6}}$ 3. $\int \frac{dx}{\sqrt{x^2 - 6x + 18}}$

Consider the following:

- In each case, let $u = x + \frac{b}{2a}$
- Use, where necessary, the formulae

  $\int \frac{u'}{u} du = \ln|u| + d$ *and* $\int \frac{dx}{\sqrt{x^2 \pm k^2}} = \ln\left|x + \sqrt{x^2 \pm k^2}\right| + d$ Where d is constant.

From activity 4.15, for the integral of the form $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$, if $a > 0$

- If $b^2 - 4ac = 0$, then

  $$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}} \int \frac{dx}{x + \frac{b}{2a}}$$ and we let $u = x + \frac{b}{2a}$

- If $b^2 - 4ac > 0$, then

  $$\int \frac{dx}{\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{a}} \int \frac{dx}{\sqrt{\left(x + \frac{b}{2a}\right)^2 + \frac{b^2 - 4ac}{4a^2}}}$$

  We let $u = x + \frac{b}{2a}$, $k^2 = \frac{b^2 - 4ac}{4a^2}$ and use the integral

  $$\int \frac{dx}{\sqrt{x^2 \pm k^2}} = \ln\left|x + \sqrt{x^2 \pm k^2}\right| + d$$

- If $b^2-4ac<0$, then

$$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{\left(x+\frac{b}{2a}\right)^2 - \frac{b^2-4ac}{4a^2}}}$$

We let $u = x+\frac{b}{2a}$, $k^2 = -\frac{b^2-4ac}{4a^2}$ and use the integral

$$\int \frac{dx}{\sqrt{x^2 \pm k^2}} = \ln\left|x+\sqrt{x^2 \pm k^2}\right| + d$$

Example 4.45

Find $\int \frac{dx}{\sqrt{x^2+2x-15}}$

Solution

$$u = x+\frac{b}{2a} = x+1 \Rightarrow du = dx,\ k^2 = \frac{b^2-4ac}{4a^2} = \frac{64}{4} = 16$$

$$\int \frac{dx}{\sqrt{x^2+2x-15}} = \int \frac{du}{\sqrt{u^2-k^2}}$$

$$= \ln\left|u+\sqrt{u^2-k^2}\right| + c \quad = \ln\left|x+1+\sqrt{(x+1)^2-16}\right| + c$$

$$= \ln\left|x+1+\sqrt{x^2+2x-15}\right| + c$$

Example 4.46

Find $\int \frac{dx}{\sqrt{2x^2+x+1}}$

Solution

$$u = x+\frac{b}{2a} = x+\frac{1}{4} \Rightarrow du = dx,\ k^2 = -\frac{b^2-4ac}{4a^2} = \frac{7}{16}$$

$$\int \frac{dx}{\sqrt{2x^2+x+1}} = \frac{1}{\sqrt{2}}\int \frac{du}{\sqrt{u^2+k^2}} \qquad = \frac{1}{\sqrt{2}}\ln\left|u+\sqrt{u^2+k^2}\right|+c$$

$$= \frac{1}{\sqrt{2}}\ln\left|x+\frac{1}{4}+\sqrt{x^2+2x+1}\right|+c$$

Notice

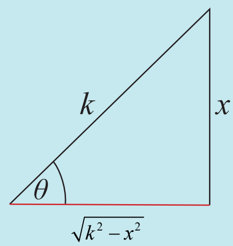
Sometimes, we will need trigonometric substitution and we can change back to the original variable afterwards.

- For integral containing $\sqrt{k^2-x^2}$, we put

  $x = k\sin\theta \Rightarrow dx = k\cos\theta\, d\theta$

  $x = k\sin\theta \Rightarrow \sin\theta = \frac{x}{k}$

From the definition of trigonometric ratios, we construct a right angle triangle whose opposite side to angled θ is x and hypotenuse is k. From Pythagoras rule, the adjacent side will be $\sqrt{k^2-x^2}$.



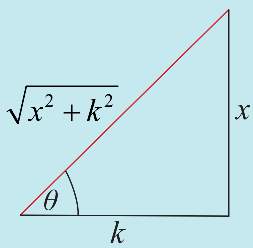
$x = k\sin\theta$

$\sqrt{k^2-x^2} = k|\cos\theta|$

For integral containing $\sqrt{x^2+k^2}$, we put $x = k\tan\theta \Rightarrow dx = k\sec^2\theta\, d\theta$

$x = k\tan\theta \Rightarrow \tan\theta = \frac{x}{k}$

From the definition of trigonometric ratios, we construct a right angled triangle whose opposite side to angle θ is x and adjacent side is k. From Pythagoras rule, the hypotenuse side will be $\sqrt{x^2+k^2}$.



$x = k\tan\theta$

$\sqrt{x^2+k^2} = k|\sec\theta|$

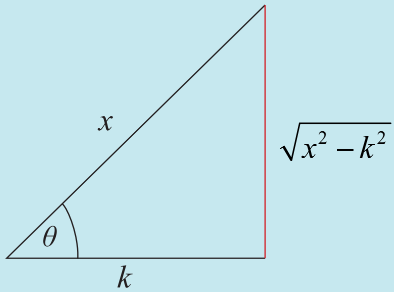
- For integral containing $\sqrt{x^2-k^2}$, we put

  $x = k\sec\theta \Rightarrow dx = k\tan\theta\sec\theta\, d\theta$

  $x = k\sec\theta \Rightarrow \sec\theta = \frac{x}{k}$

From the definition of trigonometric ratios, we construct a right angled triangle whose hypothenuse side to angle θ is x and adjacent side is k.

From Pythagoras rule, the opposite side will be $\sqrt{x^2 - k^2}$.



$$x = k \sec \theta$$

$$\sqrt{x^2 - k^2} = k \left|\tan \theta\right|$$

Example 4.47

Show that $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left|x + \sqrt{x^2 + a^2}\right| + k$, k is a constant

Solution

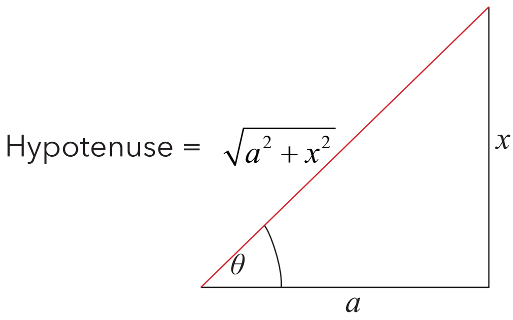
Let $x = a \tan \theta \Rightarrow dx = a \sec^2 \theta \, d\theta$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \int \frac{a \sec^2 \theta \, d\theta}{\sqrt{a^2 \tan^2 \theta + a^2}} = \int \frac{a \sec^2 \theta \, d\theta}{a\sqrt{\tan^2 \theta + 1}}$$

$$= \int \frac{\sec^2 \theta \, d\theta}{\sec \theta} = \int \sec \theta d\theta$$

$$= \ln\left|\sec \theta + \tan \theta\right| + c \qquad \left[\textit{from standard integrals}\right]$$

Recall that we made the substitution $x = a \tan \theta$



From the left triangle,

$$\tan\theta = \frac{x}{a}, \quad \sec\theta = \frac{\sqrt{a^2+x^2}}{a}$$

Then,

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right| + c$$

$$= \ln\left|\frac{\sqrt{a^2+x^2}+x}{a}\right| + c$$

$$= \ln\frac{1}{a}\left|\sqrt{a^2+x^2}+x\right| + c \qquad = \ln\left|\sqrt{a^2+x^2}+x\right| + \ln\frac{1}{a} + c$$

$$= \ln\left|\sqrt{a^2+x^2}+x\right| + k$$

Therefore, $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x+\sqrt{x^2+a^2}\right| + k$

Example 4.48

Find $\displaystyle\int x^3\sqrt{x^2-4}\,dx$

Solution

Let $x = 2\sec\theta \Rightarrow dx = 2\tan\theta\sec\theta\, d\theta$

$$\int x^3\sqrt{x^2-4}\,dx = \int 8\sec^3\theta\sqrt{4\sec^2\theta-4}\left(2\tan\theta\sec\theta\, d\theta\right)$$

$$= \int 32\sec^3\theta\sqrt{\sec^2\theta-1}\left(\tan\theta\sec\theta\, d\theta\right)$$

$$= 32\int \sec^3\theta\tan\theta\left(\tan\theta\sec\theta\, d\theta\right)$$

$$= 32\int \sec^3\theta\tan^2\theta\sec\theta\, d\theta$$

$$= 32\int \sec^4\theta\tan^2\theta\, d\theta$$

$$= 32\int \sec^2\theta \sec^2\theta \tan^2\theta \, d\theta$$

$$= 32\int \sec^2\theta\left(1+\tan^2\theta\right)\tan^2\theta \, d\theta$$

$$= 32\int \left(\tan^2\theta\sec^2\theta + \tan^4\theta\sec^2\theta\right)d\theta$$

$$= 32\int \tan^2\theta\sec^2\theta \, d\theta + 32\int \tan^4\theta\sec^2\theta \, d\theta$$

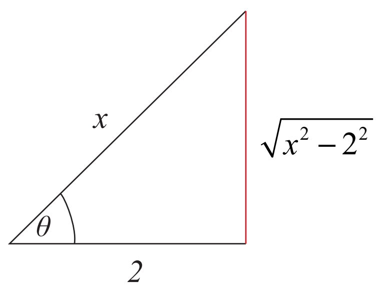
Let $u = \tan\theta \Rightarrow du = \sec^2\theta d\theta$

$$32\int \tan^2\theta\sec^2\theta \, d\theta + 32\int \tan^4\theta\sec^2\theta \, d\theta = 32\int u^2 du + 32\int u^4 du$$

$$= \frac{32u^3}{3} + \frac{32u^5}{5} + c$$

$$= \frac{32\tan^3\theta}{3} + \frac{32\tan^5\theta}{5} + c$$

Since we made the substitution $x = 2\sec\theta \Rightarrow \cos\theta = \frac{2}{x}$



From the left triangle,

$$\tan\theta = \frac{\sqrt{x^2-2^2}}{2} = \frac{\sqrt{x^2-4}}{2}$$

Then,

$$\frac{32\tan^3\theta}{3} + \frac{32\tan^5\theta}{5} = \frac{32\left(\sqrt{x^2-4}\right)^3}{3\times 8} + \frac{32\left(\sqrt{x^2-4}\right)^5}{5\times 32}$$

$$= \frac{4\left(x^2-4\right)\sqrt{x^2-4}}{3} + \frac{\left(x^2-4\right)^2\sqrt{x^2-4}}{5}$$

$$= \left(x^2-4\right)\sqrt{x^2-4}\left(\frac{4}{3} + \frac{x^2-4}{5}\right)$$

$$= \left(x^2-4\right)\sqrt{x^2-4}\left(\frac{20+3x^2-12}{15}\right)$$

$$= \frac{(x^2-4)\sqrt{x^2-4}(3x^2+8)}{15} = \frac{(3x^4-4x^2-32)\sqrt{x^2-4}}{15}$$

Therefore, $\int x^3\sqrt{x^2-4}\,dx = \frac{(3x^4-4x^2-32)\sqrt{x^2-4}}{15} + c$

Example 4.49

Find $\int \sqrt{a^2-x^2}\,dx$

Solution

Let $x = a\sin\theta \Rightarrow dx = a\cos\theta d\theta$

$$\int \sqrt{a^2-x^2}\,dx = \int \sqrt{a^2-a^2\sin^2\theta}\,(a\cos\theta\,d\theta) = \int a\sqrt{1-\sin^2\theta}\,(a\cos\theta\,d\theta)$$

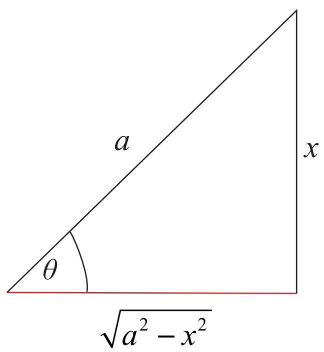
$$= a^2\int \cos\theta\cos\theta\,d\theta \quad = a^2\int \cos^2\theta\;d\theta$$

$$= \frac{a^2}{2}\int (\cos 2\theta + 1)\,d\theta, \quad \cos^2\theta = \frac{\cos 2\theta + 1}{2}$$

$$= \frac{a^2}{2}\left(\frac{\sin 2\theta}{2} + \theta\right) + c \quad = \frac{a^2}{2}\left(\frac{2\sin\theta\cos\theta}{2} + \theta\right) + c$$

$$= \frac{a^2}{2}(\sin\theta\cos\theta + \theta) + c$$

But $x = a\sin\theta \Rightarrow \sin\theta = \frac{x}{a}$



From the left triangle,

$$\cos\theta = \frac{\sqrt{a^2-x^2}}{a}, \; \sin\theta = \frac{x}{a} \Rightarrow \theta = \arcsin\frac{x}{a}$$

Then,

$$\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}(\sin\theta\cos\theta + \theta) + c$$

$$= \frac{a^2}{2}\left[\frac{x}{a}\frac{\sqrt{a^2-x^2}}{a} + \arcsin\frac{x}{a}\right] + c$$

$$= \frac{a^2}{2}\left[\frac{x}{a^2}\sqrt{a^2 - x^2} + \arcsin\frac{x}{a}\right] + c \qquad = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + c$$

Notice

For integrals of the form $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$, $a < 0$, we use

the result $\int \frac{dx}{\sqrt{k^2 - u^2}} = \arcsin\frac{x}{k} + d$

Example 4.50

Find $\int \frac{dx}{\sqrt{-x^2 + x - 1}}$

Solution

$$\int \frac{dx}{\sqrt{-x^2 - x + 1}}$$

Here,

$$-x^2 - x + 1 = -\left(x^2 + x - 1\right)$$

$$= -\left[\left(x + \frac{1}{2}\right)^2 - \frac{1+4}{4}\right] = -\left[\left(x + \frac{1}{2}\right)^2 - \frac{5}{4}\right]$$

$$= \frac{5}{4} - \left(x + \frac{1}{2}\right)^2 = \left(\frac{\sqrt{5}}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2$$

Let $u = x + \frac{1}{2} \Rightarrow du = dx$

$$\int \frac{dx}{\sqrt{-x^2 - x + 1}} = \int \frac{du}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - u^2}}$$

Using the result $\int \frac{dx}{\sqrt{k^2-u^2}} = \arcsin\frac{x}{k} + d$, we have

$$\int \frac{dx}{\sqrt{-x^2-x+1}} = \int \frac{du}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - u^2}} = \arcsin\frac{u}{\frac{\sqrt{5}}{2}} + d$$

$$= \arcsin\frac{2\left(x+\frac{1}{2}\right)}{\sqrt{5}} + d \quad = \arcsin\frac{2x+1}{\sqrt{5}} + d$$

Application activity 4.15

Find;

1. $\int \frac{dx}{\sqrt{x^2+2x+5}}$
2. $\int \frac{dx}{\sqrt{4-2x-x^2}}$
3. $\int \frac{dx}{\sqrt{x^2+4x+2}}$
4. $\int \frac{dx}{\sqrt{6x-x^2-5}}$
5. $\int \sqrt{x(1-x)}\,dx$

4.3.5. Integration by parts

Activity 4.16

Let $f(x) = (x-1)e^x$.

1. Differentiate $f(x)$ using the product rule.
2. From 1), determine the value of $\int xe^x dx$.
3. Is it true that $\int uvdx = \int udx \int vdx$?

From activity 4.16, we see that the integral of a product of two functions does not equal the product of the integrals of the two functions.

To develop a rule, we start with the product rule for differentiation:

$$\frac{d(uv)}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$$

Integrating both sides with respect to x yields

$$uv = \int v\frac{du}{dx}dx + \int u\frac{dv}{dx}dx$$

$$\Leftrightarrow \int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

Or $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

This is the formula for **integration by parts.**

To apply the integration by parts to a given integral, we must first factor its integrand into two parts.

An effective strategy is to choose for $\frac{dv}{dx}$ the most complicated factor that can readily be integrated. Then, we differentiate the other part, *u*, to find $\frac{du}{dx}$.

The following table can be used:

u	v'
Logarithmic function	Polynomial function
Polynomial function	Exponential function
Polynomial function	Trigonometric function
Exponential function	Trigonometric function
Trigonometric function	Exponential function
Inverse trigonometric function	Polynomial function

Example 4.51

Find $\int \ln x dx$

Solution

Here, we can write $\int \ln x dx = \int 1 \cdot \ln x dx$

Let $u = \ln x \Rightarrow du = \frac{1}{x}dx$ and $dv = dx \Rightarrow v = x$

Then,

$$\int \ln x \, dx = x \ln|x| - \int x \frac{dx}{x} = x \ln|x| - \int dx = x \ln|x| - x + c.$$

Example 4.52

Find $\int xe^x dx$

Solution

Let $u = x \Rightarrow du = dx$ and $dv = e^x dx \Rightarrow v = \int e^x dx = e^x$

Then,

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c.$$

Example 4.53

Find $\int \mathrm{e}^x \sin x \, dx$

Solution

Let $u = x \Rightarrow du = dx$ and $dv = \sin 2xdx \Rightarrow v = -\frac{1}{2}\cos 2x$

Then,

$$\int x \sin 2xdx = -\frac{x}{2}\cos 2x - \int -\frac{1}{2}\cos 2xdx$$

$$= -\frac{x}{2}\cos 2x + \int \frac{1}{2}\cos 2xdx = -\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x + c$$

Example 4.54

Find $\int e^x \sin 2xdx$

Solution

Let $u = \sin x \Rightarrow du = \cos xdx$ and $dv = e^x dx \Rightarrow v = e^x$

Then,

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx$$

We need to calculate $\int e^x \cos x dx$

Let $u = \cos x \Rightarrow du = -\sin x dx$ and $dv = e^x dx \Rightarrow v = e^x$

$$\int e^x \sin x dx = e^x \sin x - \int e^x \cos x dx$$

$$\int e^x \sin x dx = e^x \cos x - \left(e^x \sin x + \int e^x \sin x dx\right)$$

$$\int e^x \cos x dx = e^x \sin x - e^x \cos x - \int e^x \sin x dx$$

Clearly, we see that the original integral has reappeared on the RHS. Thus, by collecting like terms we have;

$$\int e^x \sin x dx + \int e^x \sin x dx = e^x \sin x - e^x \cos x$$

$$2\int e^x \sin x dx = e^x \sin x - e^x \cos x$$

Thus,

$$\int e^x \sin x dx = \frac{1}{2}\left(e^x \sin x - e^x \cos x\right) + c$$

Example 4.55

Find $\int \arcsin x dx$

Solution

We can write $\int \arcsin x dx$ as $\int 1 \cdot \arcsin x dx$

Let $u = \arcsin x \Rightarrow du = \frac{dx}{\sqrt{1-x^2}}$ and $dv = du \Rightarrow v = x$

$$\int \arcsin x dx = x \arcsin x - \frac{1}{2}\int \frac{x dx}{\sqrt{1-x^2}} = x \arcsin x + \sqrt{1-x^2} + c$$

Application activity 4.16

Use the method of integration by parts to find the following:

1. $\int x \cos 2x \, dx$
2. $\int x e^{3x} \, dx$
3. $\int x \sin 4x \, dx$
4. $\int x^2 \ln x \, dx$
5. $\int (2x+3) e^{2x} \, dx$
6. $\int x e^{-2x} \, dx$

Integration by reduction formulae

Activity 4.17

Let $I_m = \int x^m \cos bx\,dx$, $J_m = \int x^m \sin bx\,dx$.

Apply the method integration by parts to integral J_m to show that $bJ_m - mI_{m-1} = -x^m \cos bx$.

Knowing integral I_m, we can establish a general relation, by integration by parts, which will help us to find $I_{m-1}, I_{m-2}, I_{m-3}, ..., I_0$.

Example 4.56

Let $I_m = \int x^m e^{ax} dx$. Show that $aI_m + mI_{m-1} = x^m e^{ax}$ and hence deduce the value of I_2 and I_0.

Solution

Let $u = x^m \Rightarrow du = mx^{m-1}$ and $dv = e^{ax} dx \Rightarrow v = \frac{1}{a}e^{ax}$

Then,

$$I_m = \frac{1}{a}x^m e^{ax} - \frac{m}{a}\int x^{m-1}e^{ax}dx$$

$$I_m = \frac{1}{a}x^m e^{ax} - \frac{m}{a}I_{m-1}$$

$$I_m = \frac{1}{a}\left(x^m e^{ax} - mI_{m-1}\right)$$

Now,

From the relation $I_m = \frac{1}{a}\left(x^m e^{ax} - mI_{m-1}\right)$, we can write

$$I_2 = \frac{1}{a}\left(x^2 e^{ax} - 2I_1\right) \text{ and } I_1 = \frac{1}{a}\left(xe^{ax} - I_0\right)$$

But $I_0 = \int x^0 e^{ax} dx = \int e^{ax} dx = \frac{1}{a}e^{ax} + c$

$$I_1 = \frac{1}{a}\left(xe^{ax} - \frac{1}{a}e^{ax}\right) + c$$

$$I_2 = \frac{1}{a}\left(x^2e^{ax} - 2\left(\frac{1}{a}\left(xe^{ax} - \frac{1}{a}e^{ax}\right)\right)\right) + c$$

$$= \frac{1}{a}\left(x^2e^{ax} - \frac{2x}{a}e^{ax} + \frac{2}{a^2}e^{ax}\right) + c = \frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right) + c$$

Example 4.57

Given that $I_m = \int x^m \ln x dx$. Show that $I_m(m+1) = x^{m+1}\left(\ln x - \frac{1}{m+1}\right)$ with $m \in \mathbb{Z}\setminus\{-1\}$. Deduce the value of $\int x \ln x dx$ and $\int \ln x dx$.

Solution

Let $u = \ln x \Rightarrow du = \frac{dx}{x}$ and $dv = x^m dx \Rightarrow v = \frac{x^{m+1}}{m+1}$

$$I_m = \frac{x^{m+1}}{m+1}\ln x - \int \frac{x^{m+1}}{m+1}\frac{dx}{x} = \frac{x^{m+1}}{m+1}\ln x - \int \frac{x^m \cdot x}{m+1}\frac{dx}{x}$$

$$= \frac{x^{m+1}}{m+1}\ln x - \int \frac{x^m}{m+1}dx = \frac{x^{m+1}}{m+1}\ln x - \frac{x^{m+1}}{(m+1)(m+1)}$$

Thus,

$$I_m = \frac{x^{m+1}}{m+1}\left(\ln x - \frac{1}{m+1}\right)$$

Thus, $I_m(m+1) = x^{m+1}\left(\ln x - \frac{1}{m+1}\right)$; as required

From I_m expression, we see that;

For $m = 1$, $I_1 = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right)$ and $I_1 = \int x \ln x dx$

Then, $\int x \ln x dx = \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c$

Also,

For $m=0$, $I_0 = x(\ln x - 1)$ and $I_0 = \int \ln x dx$

Then, $\int \ln x dx = x(\ln x - 1) + c$

Application activity 4.17

Use the method of integration by parts to derive the reduction formula for:

1. $I_n = \int x^n e^{ax}\, dx$
2. $I_n = \int \tan^n x\, dx$ and hence find $\int \tan^5 x\, dx$
3. $I_n = \int \sin^n x\, dx$
4. $I_n = \int \cos^n x\, dx$
5. $I_n = \int (\ln x)^n\, dx$

4.3.6. Integration by Maclaurin series

Activity 4.18

Consider the function $f(x) = \ln(1+x)$. Find the;

1. Maclaurin polynomial for $f(x)$.
2. Integral of the polynomial obtained in 1).

For some integrals, we can use Maclaurin series of the function to help in their integration.

Example 4.58

Find by Maclaurin series $\int \sqrt[3]{1+x+x^2}\, dx$

Solution

The Maclaurin series of $\sqrt[3]{1+x+x^2}$ is $\sqrt[3]{1+x+x^2} = 1 + \frac{x}{3} + \frac{2x^2}{9} + \ldots$

Then,

$$\int \sqrt[3]{1+x+x^2}\, dx = \int \left(1 + \frac{x}{3} + \frac{2x^2}{9} + \ldots\right) dx = x + \frac{x^2}{6} + \frac{2x^3}{27} + \ldots + c$$

Example 4.59

Find by Maclaurin series $\int \ln(1-t)\,dt$

Solution

The Maclaurin series of $\ln(1-t)$ is $\ln(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \ldots$

Then,

$$\int \ln(1-t)dt = \int\left(-t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \ldots\right)dt = -\frac{t^2}{2} - \frac{t^3}{6} - \frac{t^4}{12} - \frac{t^5}{20} - \ldots + c$$

Application activity 4.18

Use Maclaurin series to find;

1. $\int e^{-3x}dx$
2. $\int \sin x dx$
3. $\int \cos x dx$
4. $\int \tan x dx$
5. $\int \sqrt{1+x}dx$

4.4. Definite integrals

4.4.1. Definition

Activity 4.19

Consider the function $f(x) = x^2 - 2x + 3$.

1. Find the indefinite integrals $F(x)$ of $f(x)$.
2. Evaluate $F(1) - F(-1)$.

We define the definite integrals of the function $f(x)$ with respect to x from a to b to be

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a);$$

where $F(x)$ is the anti-derivative of $f(x)$.

We call a and b the lower and upper limits of integration respectively.

1. The definite integral of a function $f(x)$ which lies above the x-axis can be interpreted as the **area under the curve** of $f(x)$.

2. Let us find that area enclosed by the curve $y = f(x)$ and the lines $x = a$ and $x = b$ as illustrated in figure 4.2.

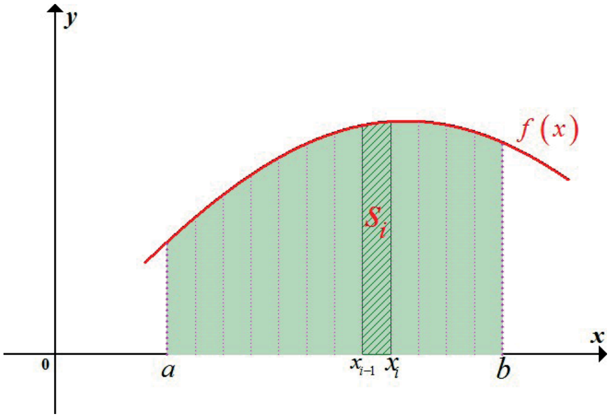


Figure 4.2. *Definite integral of a function*

The area S_i of the strip between x_{i-1} and x_i is approximately equal to the area of a rectangle with width $l = \Delta x$ and length $L = f(x_i)$ i.e. $S_i = f(x_i) \cdot \Delta x$.

The total area A is $\sum_{i=1}^{n} S_i = \sum_{i-1}^{n} f(x_i) \cdot \Delta x$ or $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x$;

this is known as **Sum of Riemann**.

If f is continuous on $[a,b]$, then $A = \int_a^b f(x)\,dx$.

Notice

Integration constants are not written in definite integrals since they always cancel out.

Consider; $\int_a^b f(x)dx = [F(x)]_a^b$

$= (F(b) + c) - (F(a) + c) = F(b) + c - F(a) - c$

$= F(b) - F(a)$

Remark: $\int_a^a f(x)dx = 0$ and $\int_a^a 0dx = 0$

Example 4.60

Determine the value of the definite integral $\int_1^2 x^3 dx$

Solution

First, we calculate $\int x^3 dx = \frac{x^4}{4}$

Then, $\int_1^2 x^3 dx = \left[\frac{x^4}{4}\right]_1^2 = \frac{2^4}{4} - \frac{1^4}{4} = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$

Therefore, $\int_1^2 x^3 dx = \frac{15}{4}$

Example 4.61

Evaluate $\int_0^1 x^2 dx$

Solution

$\int x^2 dx = \frac{x^3}{3}$

Then, $\int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$

Example 4.62

Work out $\int_{-2}^2 |x-1|\, dx$

Solution

Since $|x-1| = x-1 \ \ for \ \ x \geq 1$ and $|x-1| = -x+1 \ \ for \ \ x \leq 1$, then

$$\int_{-2}^2 |x-1|\, dx = \int_{-2}^1 (-x+1)\, dx + \int_1^2 (x-1)\, dx$$

$$= \left[-\frac{x^2}{2} + x\right]_{-2}^1 + \left[\frac{x^2}{2} - x\right]_1^2$$

$$= \left(-\frac{1}{2} + 1\right) - (-2-2) + (2-2) - \left(\frac{1}{2} - 1\right) = 5$$

Example 4.63

Evaluate $\int_0^3 |3x-5|\,dx$

Solution

As $|3x-5| = \begin{cases} -3x+5, & x < \frac{5}{3} \\ 3x-5, & x \ge \frac{5}{3} \end{cases}$, we split the integral into parts

at the point where $|3x-5|$ changes from $-3x+5$ to $3x-5$, namely at $x = \frac{5}{3}$, then

$$\int_0^3 (3x-5)\,dx = \int_0^{\frac{5}{3}} (-3x+5)\,dx + \int_{\frac{5}{3}}^3 (3x-5)\,dx$$

$$= \left[-\frac{3}{2}x^2 + 5x\right]_0^{\frac{5}{3}} + \left[\frac{3}{2}x^2 - 5x\right]_{\frac{5}{3}}^3$$

$$= -\frac{3}{2}\left(\frac{5}{3}\right)^2 + 5\left(\frac{5}{3}\right) - 0 + \left[\frac{3}{2}(3)^2 - 5(3)\right] - \left[\frac{3}{2}\left(\frac{5}{3}\right)^2 - 5\left(\frac{5}{3}\right)\right]$$

$$= \frac{25}{6} - \frac{3}{2} + \frac{25}{6} = \frac{25}{3} - \frac{3}{2} = \frac{41}{6}$$

Example 4.64

Evaluate $\int_0^4 |x^3 - 5x^2 + 6x|\,dx$

Solution

We split up our integral depending on where $x^3 - 5x^2 + 6x$ is non-negative

$$x^3 - 5x^2 + 6x = x(x^2 - 5x + 6) = x(x-2)(x-3)$$

Table of sign:

x	0	2		3	
x	+	+	+		+
$(x-2)(x-3)$	+	0	−	0	+
$x^3-5x^2+6x=x(x-2)(x-3)$	+	0	−	0	+

From the table of sign, we get

$$\left|x^3-5x^2+6x\right|=\begin{cases}-\left(x^3-5x^2+6x\right), if\ x\in\left]-\infty,0\right[\cup\left]2,3\right[\\ x^3-5x^2+6x,\ if\ x\in[0,2]\cup[3,+\infty[\end{cases}$$

Now, we can integrate, using definition of absolute value

$$\int_0^4\left|x^3-5x^2+6x\right|dx=$$

$$x\big|\,dx=\int_0^2\left(x^3-5x^2+6x\right)dx+\int_2^3-\left(x^3-5x^2+6x\right)dx+\int_3^4\left(x^3-5x^2+6x\right)dx$$

$$=\frac{x^4}{4}-\frac{5}{3}x^3+3x^2\Big|_0^2-\left(\frac{x^4}{4}-\frac{5}{3}x^3+3x^2\right)\Big|_2^3+\left(\frac{x^4}{4}-\frac{5}{3}x^3+3x^2\right)\Big|_3^4$$

$$=\frac{8}{3}-0-\left(\frac{9}{4}-\frac{8}{3}\right)+\left[\left(112-\frac{320}{3}\right)-\frac{9}{4}\right]$$

$$=\frac{8}{3}+\frac{5}{12}+\left(\frac{16}{3}-\frac{9}{4}\right)\ =\frac{8}{3}+\frac{5}{12}+\frac{37}{12}\ =\frac{74}{12}\ =\frac{37}{6}$$

Application activity 4.19

Evaluate the integrals;

1. $\int_0^3 xdx$
2. $\int_1^2 (x^2-x)\,\mathrm{dx}$
3. $\int_1^2 (3x^2-6x)dx$
4. $\int_{-1}^2\left(x^3+3x^2-4\right)dx$
5. $\int_0^3\left|2x^2-8\right|dx$
6. $\int_0^{2\pi}\left|\sin x\right|dx$
7. $\int_0^{10}\left|x-5\right|dx$
8. $\int_1^5\left|-2x^3+24x\right|dx$

4.4.2. Properties of definite integrals

Activity 4.20

Consider the function $f(x)=x^2$.

1. Determine $\int_{-3}^{0} f(x)dx$ and $\int_{0}^{-3} f(x)dx$. Give your observation.
2. Also, obtain $\int_{-2}^{2} x^2dx$ and $\int_{-2}^{0} x^2dx+\int_{0}^{2} x^2dx$. Give your observation.

From activity 4.20, we remark the following:

1. **Permutation of bounds**: If $f(x)$ is defined on (a,b) except may be at a finite number of points, then $\int_a^b f(x)dx=-\int_b^a f(x)dx$.

2. **Chasles relation**: For any arbitrary numbers a and b, and any $c\in[a,b]$

$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$$

3. **Positivity**: Let f be a continuous function on interval $I=[a,b]$; the elements of I

If $f\geq 0$ on I and if $a\leq b$, then $\int_a^b f(x)dx\geq 0$

Also, if $f(x)\leq g(x)$ on $[a,b]$, then $\int_a^b f(x)dx\leq\int_a^b g(x)dx$.

Theorem 4.1: Mean value theorem

Let f be a continuous function on interval $[a,b]$, there exists a number $c\in[a,b]$ such that $f(c)=\frac{1}{b-a}\int_a^b f(x)dx$.

This value $\underline{f(c)}$ is called the **average value** of $f(x)$ on $[a,b]$ and is denoted as $\overline{f(x)}$.

Example 4.65

Find the average value of $f(x)=\sin x$ on $[0,\pi]$

Solution

$$\overline{f(x)} = \frac{1}{\pi - 0}\int_0^{\pi} \sin x dx = \frac{1}{\pi}[-\cos x]_0^{\pi} = -\frac{1}{\pi}[\cos x]_0^{\pi} = -\frac{1}{\pi}(-1-1) = -\frac{1}{\pi}(-2) = \frac{2}{\pi}$$

Example 4.66

Show that: $\int_1^4 \frac{x}{2}dx = \int_1^2 \frac{x}{2}dx + \int_1^4 \frac{x}{2}dx$

LHS= $\int_1^4 \frac{x}{2}dx = \left[\frac{x^2}{4}\right]_1^4 = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$

RHS= $\int_1^2 \frac{x}{2}dx + \int_1^4 \frac{x}{2}dx = \left[\frac{x^2}{4}\right]_1^2 + \left[\frac{x^2}{4}\right]_2^4 = \frac{4}{4} - \frac{1}{4} + \frac{16}{4} - \frac{4}{4} = \frac{16}{4} - \frac{1}{4} = \frac{15}{4}$

Since RHS=LHS= $\frac{15}{4}$ as required

> **Notice**
>
> **Techniques of integration**
>
> The methods of integration of definite integrals are the same as for indefinite integrals but in changing the variable remember to change the bounds.

Example 4.67

Evaluate: $I = \int_0^{-4} \frac{xdx}{\sqrt{1+3x^2}}$

Solution

Let $1+3x^2 = t^2 \Rightarrow 6xdx = 2tdt \Rightarrow xdx = \frac{t}{3}dt$

If $x \to -4, t \to 7$ also if $x \to 0, t \to 1$

$$I = \int_0^{-4} \frac{xdx}{\sqrt{1+3x^2}} = \int_1^7 \frac{t\,dt}{3t} = \frac{1}{3}\int_1^7 dt = \frac{1}{3}[t]_1^7 = \frac{1}{3}(7-1) = 2$$

Or we can say, without changing the bounds,

$$I = \frac{1}{3}\int_1^7 dt = \frac{1}{3}[t]_1^7$$

But $t=\sqrt{1+3x^2}$

Therefore, $I=\frac{1}{3}\left[\sqrt{1+3x^2}\right]_0^{-4}=\frac{1}{3}(7-1)=2$

Application activity 4.20

Evaluate:

1. $\int_0^{2\pi}|\cos x|\,dx$
2. $\int_{-1}^{2}x|x|\,dx$
3. $\int_0^1\frac{\arctan x}{1+x^2}dx$
4. $\int_0^1\arcsin x\,dx$

4.4.3. Improper integrals

The definite integral $\int_a^b f(x)dx$ is called an **improper integral** if one of two situations occurs:

- The limit a or b (or both the bounds) are infinites.
- The function $f(x)$ has one or more points of discontinuity in the interval $[a,b]$.

Infinite limits of integration

Activity 4.21

Evaluate the integrals:

1. $\lim\limits_{n\to+\infty}\int_0^n\frac{dx}{x^2+4}$
2. $\lim\limits_{n\to-\infty}\int_n^{-4}\frac{xdx}{\sqrt{1+3x^2}}$

Let $f(x)$ be a continuous function on the interval $[a,+\infty[$.

Then, we define the improper integral as $\int_a^{+\infty}f(x)dx=\lim\limits_{n\to+\infty}\int_a^n f(x)dx$

considering the case when $f(x)$ is a continuous function on the interval $]-\infty,b]$; then, we define the improper integral

as $\int_{-\infty}^{b}f(x)dx=\lim\limits_{n\to-\infty}\int_n^b f(x)dx$.

If these limits exist and are finite, then we say that the improper integrals are **convergent;** otherwise the integrals are **divergent.**

Let $f(x)$ be a continuous function for all real numbers. By Chasles theorem, we can write $\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx$.

If for real number c, both of the integrals in the right hand side are convergent, then we say that the integral $\int_{-\infty}^{+\infty} f(x)dx$ is also **convergent**; otherwise it is **divergent.**

Theorem 4.2. Comparison theorems

Let $f(x)$ and $g(x)$ be continuous on the interval $[a,+\infty[$. Suppose that $0 \le g(x) \le f(x)$ for all x in the interval $[a,+\infty[$,

- If $\int_{a}^{+\infty} f(x)dx$ is **convergent,** then, $\int_{a}^{+\infty} g(x)dx$ is also **convergent.**
- If $\int_{a}^{+\infty} g(x)dx$ is **divergent,** then, $\int_{a}^{+\infty} f(x)dx$ is also **divergent.**
- If $\int_{a}^{+\infty} |f(x)|dx$ is **convergent,** then, $\int_{a}^{+\infty} f(x)dx$ is also **convergent**. In this case, we say that the integral $\int_{a}^{+\infty} f(x)dx$ is **absolutely convergent.**

Example 4.68

Evaluate $\int_{0}^{+\infty} \frac{dx}{x^2+16}$

Solution

$$\int_{0}^{+\infty} \frac{dx}{x^2+16} = \lim_{n\to+\infty} \int_{0}^{n} \frac{dx}{x^2+16}$$

$$= \lim_{n\to+\infty} \left[\frac{1}{4}\arctan\frac{x}{4}\right]_0^n = \frac{1}{4}\lim_{n\to+\infty}\left(\arctan\frac{n}{4} - \arctan 0\right) = \frac{1}{4}\lim_{n\to+\infty}\arctan\frac{n}{4}$$

$$= \frac{1}{4}\times\frac{\pi}{2} = \frac{\pi}{8}$$

Hence, the integral converges to $\frac{\pi}{8}$.

Example 4.69

Determine whether the integral $\int_{1}^{+\infty} \frac{dx}{x^2 e^x}$ converges or diverges.

Solution

Note that $\frac{1}{x^2e^x} \le \frac{1}{x^2}$ for all values $x \ge 1$

Since the improper integral $\int_1^{+\infty} \frac{dx}{x^2}$ is convergent as

$$\int_1^{+\infty} \frac{dx}{x^2} = \lim_{n\to+\infty} \int_1^n \frac{dx}{x^2}$$

$$= \lim_{n\to+\infty} \int_1^n +x^{-2}dx = \lim_{n\to+\infty} \left[\frac{x^{-1}}{-1}\right]_1^n = \lim_{n\to+\infty} \left[-\frac{1}{x}\right]_1^n = \lim_{n\to+\infty} \left[-\frac{1}{n}+1\right] = 1$$

then, the given integral $\int_1^{+\infty} \frac{dx}{x^2e^x}$ is also **convergent** by comparison theorems.

Application activity 4.21

Determine whether each of the following improper integral converges or diverges.

1. $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$
2. $\int_0^{\infty} e^{-2x}\, dx$
3. $\int_{-\infty}^{-1} \frac{dx}{x^2+1}$
4. $\int_2^{\infty} \frac{dx}{(x-1)^3}$
5. $\int_3^{\infty} \frac{dx}{(2x-1)^{\frac{2}{3}}}$
6. $\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2}dx$

Discontinuous integrand

Activity 4.22

Determine the points of discontinuity of the following functions in the given interval I.

1. $f(x) = \frac{x}{x-1}, \quad I = [0,4]$
2. $f(x) = \frac{2}{x^2-3x-10}, \quad I = [-10,0]$
3. $f(x) = \frac{3x+1}{\ln x}, \quad I = \left[\frac{1}{2},3\right]$

Let $f(x)$ be a function which is continuous on the interval $[a,b[$ and discontinuous at $x=b$. Then we define the improper integral as $\int_a^b f(x)dx = \lim_{t\to b}\int_a^t f(x)dx$.

Similarly, if the function $f(x)$ is continuous on the interval $]a,b]$ and discontinuous at $x=a$, then we can write

$$\int_a^b f(x)dx = \lim_{t\to a}\int_t^b f(x)dx.$$

If these limits exist and are finite, then we say that the integrals are **convergent**; otherwise the integrals are **divergent.**

Let $f(x)$ be a continuous function for all real numbers x in the interval $]a,b[$, except for some point $c\in(a,b)$, then,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx = \lim_{t\to c}\int_a^t f(x)dx + \lim_{t\to c}\int_t^b f(x)dx$$

We say that the integral $\int_a^b f(x)dx$ is **convergent** if both of the

integrals in the right hand side are also convergent. Otherwise the improper integral is **divergent**.

Example 4.70

Evaluate $\int_{-2}^{2}\frac{dx}{x^3}$

Solution

Since there is a discontinuity at $x=0$, we must consider two improper integrals;

$$\int_{-2}^{2}\frac{dx}{x^3} = \int_{-2}^{0}\frac{dx}{x^3} + \int_{0}^{2}\frac{dx}{x^3}$$

$$\int_{-2}^{2}\frac{dx}{x^3} = \lim_{t\to 0}\int_{-2}^{t}\frac{dx}{x^3} + \lim_{t\to 0}\int_{t}^{2}\frac{dx}{x^3} = \lim_{t\to 0}\left[\frac{x^{-2}}{-2}\right]_2^t + \lim_{t\to 0}\left[\frac{x^{-2}}{-2}\right]_t^2$$

$$= \lim_{t\to 0}\left[-\frac{1}{2x^2}\right]_{-2}^{t} + \lim_{t\to 0}\left[-\frac{1}{2x^2}\right]_t^2$$

Now considering the term

$$\lim_{t\to 0}\left[-\frac{1}{2x^2}\right]_{-2}^{t} = -\frac{1}{2}\lim_{t\to 0}\left[\frac{1}{x^2}\right]_{-2}^{t} = -\frac{1}{2}\lim_{t\to 0}\left(\frac{1}{t^2}-\frac{1}{4}\right) = \infty$$

Since it is divergent, the initial integral also **diverges**.

Application activity 4.22

1. Determine whether the integral $\int_0^4 \frac{dx}{(x-2)^3}$ converges or diverges.
2. Evaluate and comment on your answer in each case.

   a) $\int_0^1 (1-x)^{-\frac{2}{3}}dx$ b) $\int_0^2 \frac{dx}{x}$

   c) $\int_1^4 \frac{dx}{(x-2)^2}$ d) $\int_{-2}^1 \frac{dx}{x^{\frac{4}{5}}}$

4.5. Applications

4.5.1. Calculation of area of a plane surface

Activity 4.23

Consider the function $f(x)=x$.

1. Sketch the curve of $f(x)$ on the $xy-plane$.
2. Shade the region enclosed by the curve of the given function and the $x-axis$ for $0<x<4$.
3. Using the formula for finding the area of plane figures, find the area of the region you shaded in 2).
4. Find the definite integral of the given function for $0<x<4$.
5. Comment on your results in 3) and 4).

The definite integral of a function $f(x)$ which lies above the x-axis denotes the area under the curve of $f(x)$ as shown in figure 4.3.

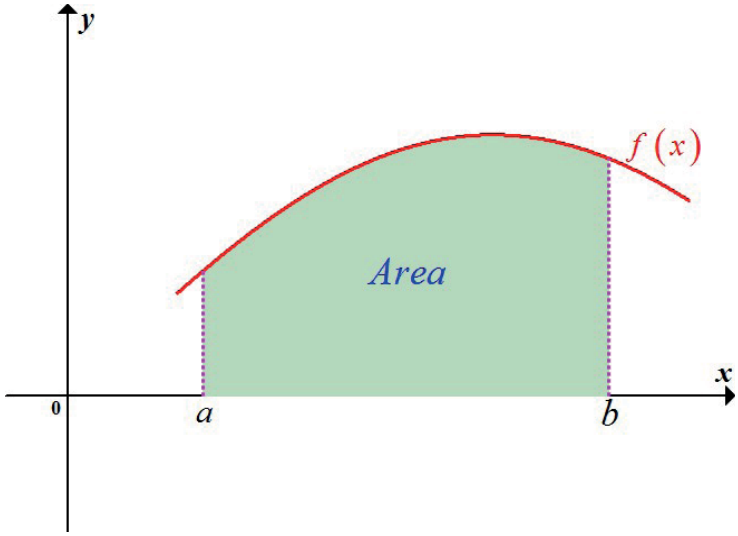


Figure 4.3. *Area enclosed by a curve of a function and x-axis*

Given function $f(x)$ which lies above the x-axis, the area enclosed by the curve of $f(x)$ and the x-axis in interval $[a,b]$ is given by

$$A = \int_a^b f(x)\,dx$$

The area between two curves

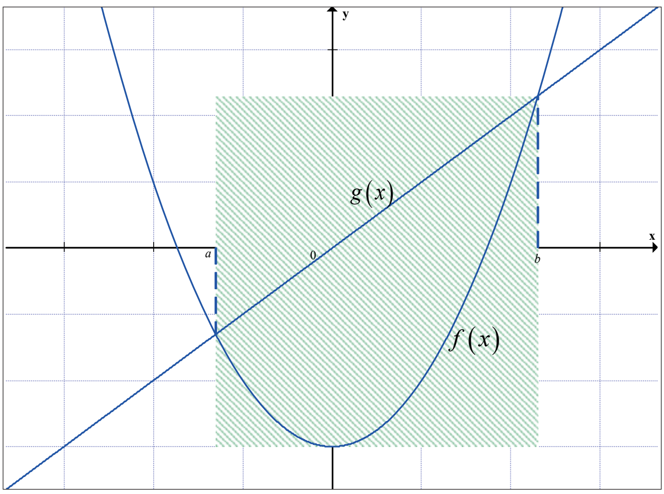


Figure 4.4. *Area between two curves*

The area between two functions $f(x)$ and $g(x)$ where $f(x) \le g(x)$ in $[a,b]$ is given by

$$\int_a^b [g(x) - f(x)]dx = \int_a^b g(x)dx - \int_a^b f(x)dx$$

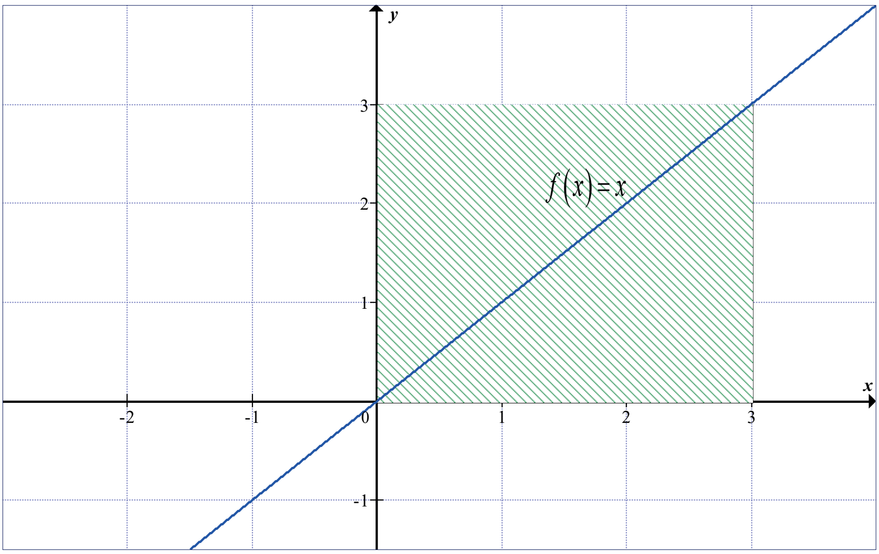
as illustrated in figure 4.4.

Example 4.71

Find the area enclosed by $x-axis$ and the first bisector.

Solution

$x-axis$ is represented by the function $g(x)=0$ and the first bisector is represented by $f(x)=x$.



The area is given by

$$A=\int_0^3 (0-x)\,dx=\left[\frac{x^2}{2}\right]_0^3=\frac{3^2}{2}-\frac{0}{2}=\frac{9}{2}\,sq.units.$$

Alternative method

From the figure, the shaded area is a triangle with base 3 units and height 3 units. So, the area is

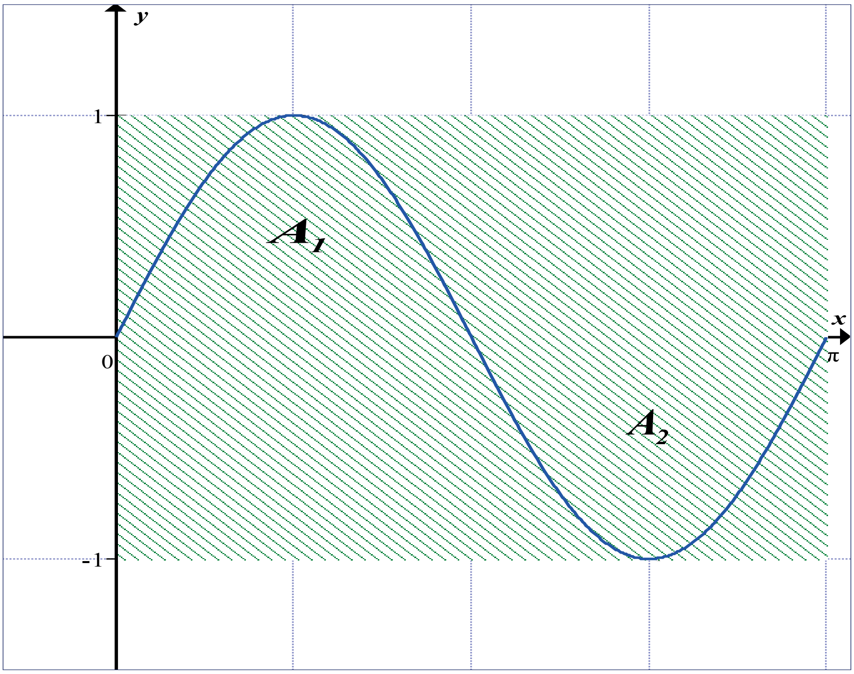
$$A=\frac{1}{2}base\times height=\frac{9}{2}\,sq.units.$$

Example 4.72

Find the area of a sinusoid in $[0,2\pi]$.

Solution

The sketch of the two curves is as shown below:



For A_1, we have two functions $g(x)=\sin x$ and $f(x)=0$.

Then, $A_1=\int_0^{\pi}(\sin x-0)\,dx=\int_0^{\pi}\sin x\,dx$

$$=[-\cos x]_0^{\pi}=-\cos\pi+\cos 0=2\,sq.\ units$$

For A_2, we have $g(x)=0$ and $f(x)=\sin x$.

Then, $A_2=\int_{\pi}^{2\pi}(0-\sin x)dx$

$$=\int_{\pi}^{2\pi}-\sin x dx=[\cos x]_{\pi}^{2\pi}=\cos 2\pi-\cos\pi=2\,sq.units$$

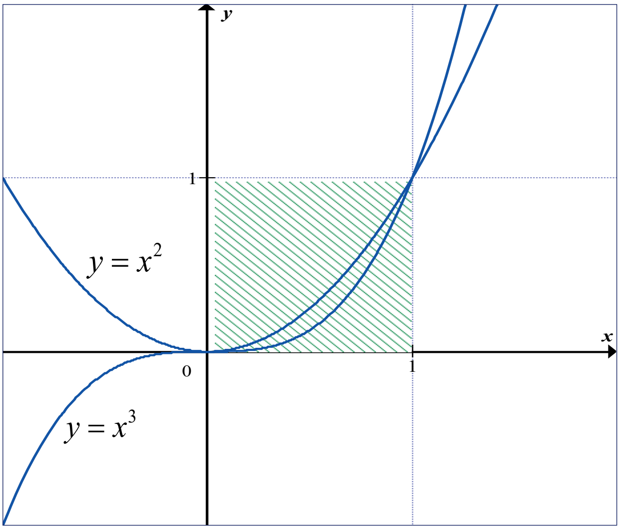
The total area is $4\,sq.\ units$.

Example 4.73

Find the area enclosed by the curves $y = x^3$ and $y = x^2$

Solution

The sketch of the two curves is as shown below:



We now need to know the intersection points of the two curves.

To do this, we solve for $x^2 = x^3$ or $x^2 - x^3 = 0$

$\Rightarrow x = 0 \;\; or \;\; x = 1$

The area is given by

$$A = \int_0^1 \left(x^2 - x^3\right) dx = \left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \;\; sq.\ units$$

Notice

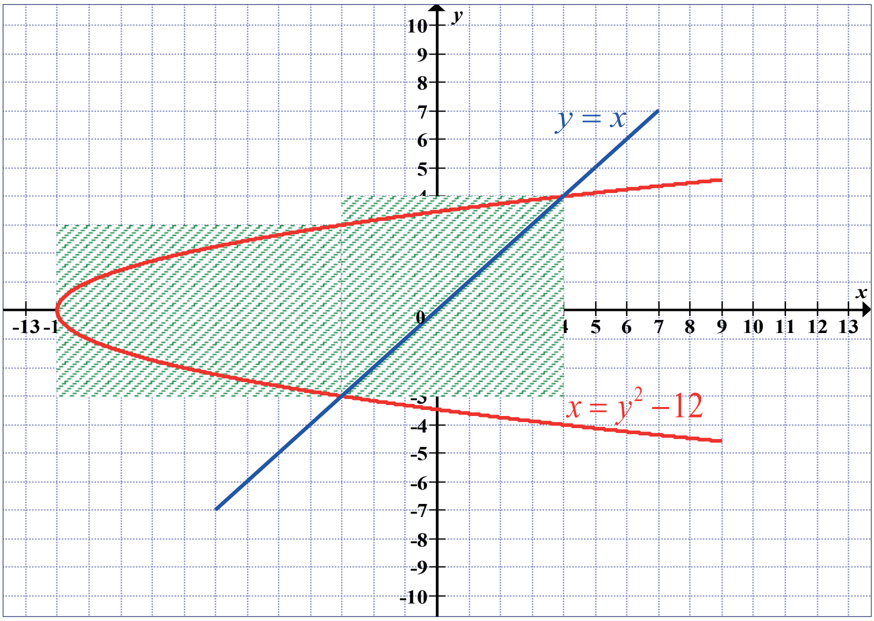
If u and v are continuous functions and if $u(y) \geq v(y)$ for all y on $[c, d]$, then the area of the region bound on the left and right by the curves $x = u(y)$, $x = v(y)$, above and below by the lines $y = d$, $y = c$ is

$$A = \int_c^d \left[v(y) - u(y)\right] dy$$

Example 4.74

Find the area of the region enclosed by the curves $x = y^2 - 12$ and $x = y$.

Solution



For the intersections of the curves

$$y^2 - 12 = x = y \Rightarrow y^2 - y - 12 = 0$$

$\Leftrightarrow (y-4)(y+3) = 0$ so $y = 4$ or $y = -3$.

Observe that $y^2 - 12 \le y$ for $-3 \le y \le 4$. Thus, the area is

$$A = \int_{-3}^{4} \left[y - \left(y^2 - 12 \right) \right] dy = \left(\frac{y^2}{2} - \frac{y^3}{3} + 12y \right) \Bigg|_{-3}^{4}$$

$$= \left(8 - \frac{64}{3} + 48 \right) - \left(\frac{9}{2} + 9 - 36 \right) = \frac{104}{3} + \frac{45}{2} = \frac{343}{6} \, sq.units$$

We would have to split the region into two parts because the equation of the lower boundary changes at $x = -3$. Then,

$$A = \int_{-12}^{-3} \left[\sqrt{12+x} - \left(-\sqrt{12+x} \right) \right] dx + \int_{-3}^{4} \left(\sqrt{12+x} - x \right) dx$$

However, if we integrate with respect to y, no splitting is necessary.

Application activity 4.23

1. Calculate the area enclosed by the curve and the straight line in each of the following:

   a) $y=(2x-1)(2x+1)$ and $x-axis$

   b) $y=x(x-1)(x-2)$ and $x-axis$

   c) $y=x^2-3x-4$ and $y=x+1$

   d) $y^2=x,\ y=\frac{1}{2}x$

   e) $y^2=4x,\ y=2x-4$

2. Find the area enclosed between the curve $y=2a^2x^2-x^4, a>0$, and the line joining its local maxima.

3. Find the area enclosed between the curves $y=-x^3+6x^2+2x-3$ and $y=(x-3)^2$

4. Find the area bound by the curve $y=\frac{1}{x^2}$, the lines $y=-27x$ and $y=-\frac{1}{8}x$.

5. Determine the total area enclosed between the curves $y=\sin x$ and $y=\cos x$ from $x=0$ to $x=2\pi$.

6. Sketch the region enclosed by the curves and find its area.

   a) $x=y^2-4y, x=0, y=0, y=4$

   b) $y^2=-x, y=x-6, y=-1, y=4$

4.5.2. Calculation of volume of a solid of revolution

Activity 4.24

1. Consider the line $y=2$ for $0\le x\le 3$.

   a) Plot the line and shade the region enclosed by the curve $y=2$ for $0\le x\le 3$ and $x-axis$.

   b) If the line $y=2$ is rotated about the $x-axis$ we obtain a solid of revolution.

Shade the region for which the area in (a) is rotated 360^0 (one revolution) about the x-axis.

c) Identify the type (nature) of solid of revolution obtained in (b) and hence determine its volume.

d) Let the area shown in (a) be divided into a number of strips parallel to y-axis and length y.

When the area is rotated 360^0 about the x-axis, each strip produces a solid of revolution approximating to a circular disc of radius y and thickness δx.

Since the volume of each disc is given by
$V(circular\ cross-sectional\ area)\ (thickness) = (\pi y^2)(\delta x)$,
then the total volume, V, between $x = 0$ and $x = 3$ is given by:

Volume, $V = \lim\limits_{\delta x \to 0} \sum\limits_{x=0}^{x=3} \pi y^2 \delta x = \int_0^3 \pi y^2\, dx$

Using the above formula, determine the total volume of the solid of revolution formed when the line $y = 2$ is rotated 360^0 about the x-axis between the limits $x = 0$ to $x = 3$.

e) Compare the results obtained in (c) and (d).

2. Repeat steps a) to d) in 1) when $y = 2x$ for $0 \le x \le 5$.

From activity 4.24, by considering function $f(x)$, the volume of the solid of revolution bound by the curve of the function $f(x)$ about the $x-axis$ calculated from $x = a$ to $x = b$, is given by $V = \pi\int_a^b f^2(x)dx$.

This method is called **disc method.**

Consider a region of $f(x)$ between $x = b$ and $x = b$ revolving around $x-axis$ as illustrated in figure 4.5. The volume of the solid of revolution is obtained by considering the area $A(x)$ of the disc of radius $y = f(x)$ such that

$A(x) = \pi y^2$ and

$$V = \int_a^b \pi y^2 dx = \int_a^b \pi f^2(x)\,dx = \pi\int_a^b f^2(x)\,dx .$$

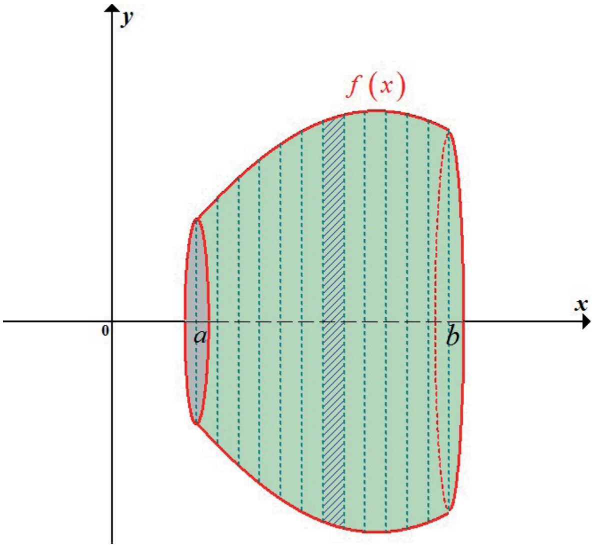


Figure 4.5. *Volume of revolution*

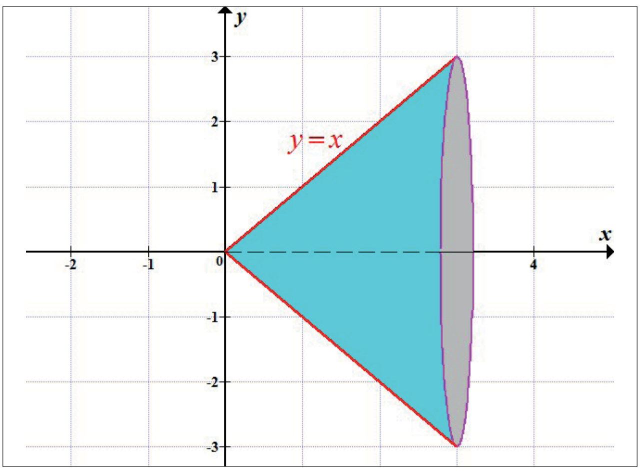
When using this method, it is necessary to integrate along the axis of revolution.

If the region is rotated about a horizontal line, integrate with respect to x, and if the region is revolved about a vertical line, integrate with respect to y.

Example 4.75

Use integration to find the volume of the solid generated when the line $y = x$ for $0 \le x \le 3$ is rotated through one revolution (360°) about the $x - axis$.

Solution



$$V = \pi\int_0^3 y^2\,dx = \pi\int_0^3 x^2\,dx = \pi\left[\frac{x^3}{3}\right]_0^3 = \pi\frac{3^3}{3} = 9\pi \ cubic\,units$$

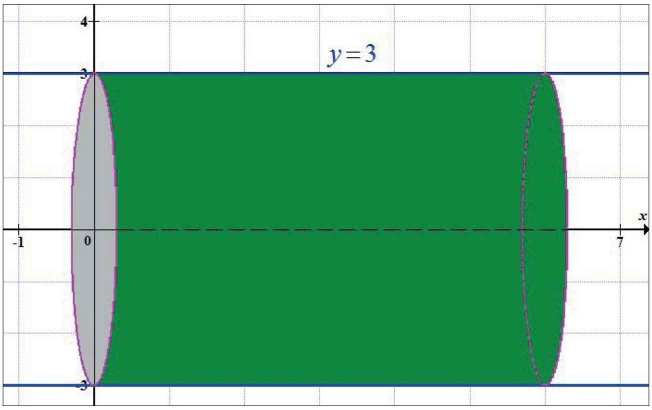
Note that the above figure shows a cone with radius 3 and height 3. Thus, we could find the volume of the cone formed using the formula $\frac{\pi}{3}r^2h$.

So, $V = \frac{\pi}{3}\times 3^2 \times 3 = 9\pi \ cubic\ units$ (as before).

Example 4.76

Use integration to find the volume of the solid generated when the line $y = 3$ for $0 \le x \le 6$ is revolved around the $x-axis$.

Solution



$$V = \pi\int_0^6 y^2\,dx = \pi\int_0^6 (3)^2\,dx = \pi[9x]_0^6 = \pi(54) = 54\pi\,cubic\,units.$$

Note that the above figure is a **cylinder** with radius 3 and height 6. Then, we could find the volume of the cylinder using the formula $\pi r^2 h$.

So, $V = \pi\times 3^2 \times 6 = 54\pi \ cubic\ units.$

Example 4.77

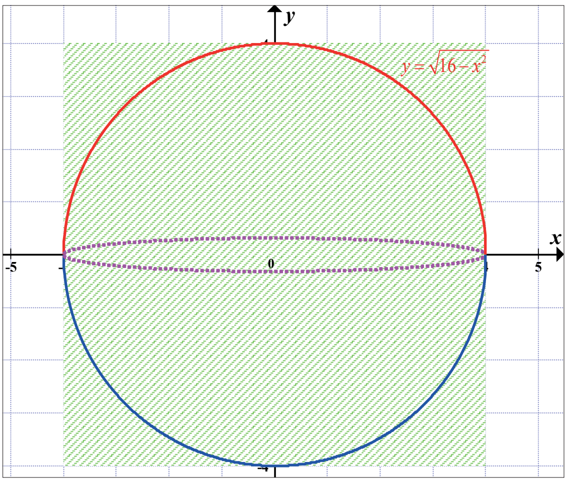
Use integration to find the volume of the solid generated when a half circle with centre $(0,0)$ and radius 4 is revolved around the $x-axis$.

Solution

Recall that the circle with centre $(0,0)$ and radius 4 is given by $x^2+y^2=16$.

Using the equation, we can write $y=\pm\sqrt{16-x^2}$

We need to use the positive part; $y=\sqrt{16-x^2}$ as shown in the following figure.



From the above figure, we will integrate from -4 to 4

$$V=\pi\int_{-4}^{4} y^2\,dx=\pi\int_{-4}^{4}\left(\sqrt{16-x^2}\right)^2 dx$$

$$=\pi\left[16x-\frac{x^3}{3}\right]_{-4}^{4}=\pi\left(64-\frac{64}{3}+64-\frac{64}{3}\right)=\pi\left(\frac{192-64+192-64}{3}\right)$$

$$=\frac{256}{3}\pi \text{ cubic units.}$$

Note that the above figure is a **sphere** with radius 4. Then, we may find the volume of the sphere formed using the formula $\frac{4}{3}\pi r^3$.

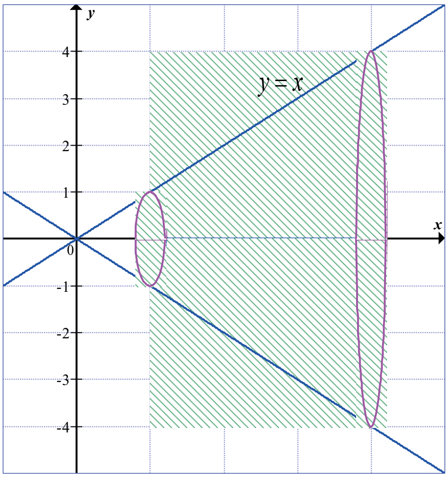
So, $V=\frac{4}{3}\pi(4)^3=\frac{256}{3}\pi$ *cubic units*.

Example 4.78

Use integration to find the volume of the solid generated when the line $y=x$ for $1\le x\le 4$ is revolved around the $x-axis$.

Solution

Let us consider the figure below;



$$V = \pi \int_a^b y^2 dx$$

$$= \pi \int_1^4 x^2 dx$$

$$= \pi \left[\frac{x^3}{3} \right]_1^4$$

$$= \pi \left(\frac{64}{3} - \frac{1}{3} \right)$$

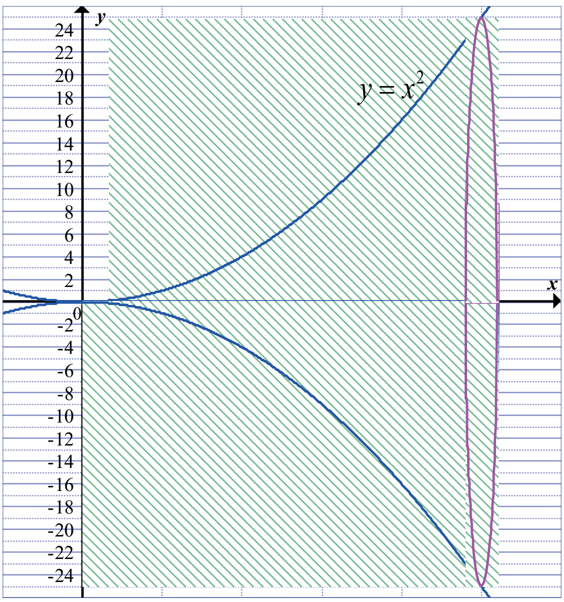
$$= 21\pi \quad cubic\ units$$

Example 4.79

Find the volume of the solid revolution formed when the area closed by the curve $y = x^2$ for $0 \le x \le 5$ is revolved about the x-axis.

Solution

Consider the figure below;



Volume is

$$V = \pi \int_a^b f^2 dx$$

$$= \pi \int_0^5 \left(x^2 \right)^2 dx$$

$$= \pi \int_0^5 x^4 dx$$

$$= \pi \left[\frac{x^5}{5} \right]_0^5$$

$$= \pi \left(625 - 0 \right)$$

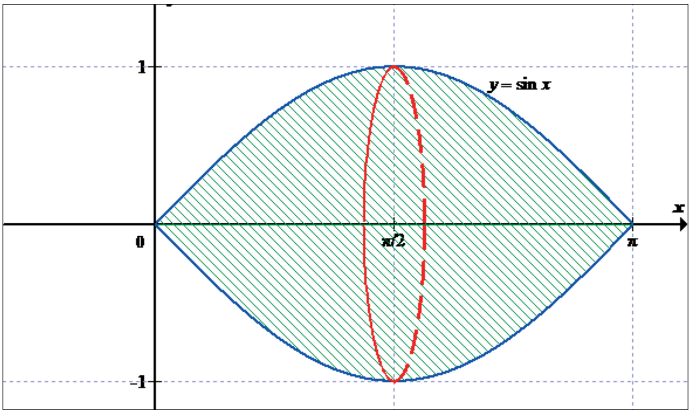
$$= 625\pi \quad cubic\ units$$

Example 4.80

Suppose one arch of $y = \sin x$ is rotated about the x-axis. What is the volume of the solid revolution formed?

Solution

Consider the figure below;



$$V = \pi \int_a^b y^2 dx$$

$$= \pi \int_0^\pi \sin^2 x dx \; = \pi \int_0^\pi \frac{1}{2}(1-\cos 2x)dx \left[\sin^2 x = \frac{1}{2}(1-\cos 2x)\right]$$

$$= \frac{\pi}{2}\int_0^\pi (1-\cos 2x)dx \; = \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^\pi \; = \frac{\pi}{2}\left[\pi - \frac{1}{2}\sin 2\pi - 0 + \frac{1}{2}\sin 0\right)$$

$$= \frac{\pi}{2}[\pi] = \frac{\pi^2}{2} \; \textit{cubic units}$$

Notice

Since $f(x)$ is continuous and strictly increasing over the interval $[a, b]$, then the inverse function $x = f^{-1}(y)$ is strictly increasing over the interval $[f(a), f(b)]$. Hence, the volume generated by rotating the region R about the $y-axis$ is

$$V = \pi \int_{f(a)}^{f(b)} x^2 dy$$

Volume for two defining functions (Washer method)

The inner radius of the solid formed is the distance from the axis of revolution to the edge of the region closest to the axis of revolution, and the outer radius of the solid formed is the distance from the axis of revolution to the edge of the region farthest from the axis of revolution.

If the region bound by outer radius $y_U = g(x)$ (on top) and inner radius $y_L = f(x)$ and the lines $x = a$, $x = b$ is revolved about $x-axis$, the the volume of the solid of revolution is given by:

$$V = \pi\int_a^b \left([g(x)]^2 - [f(x)]^2\right)dx$$

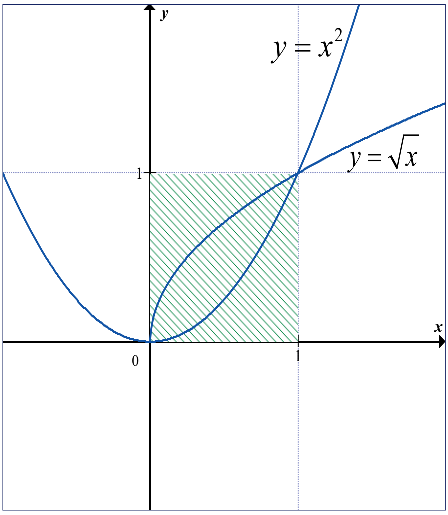
This method is called **washer method.**

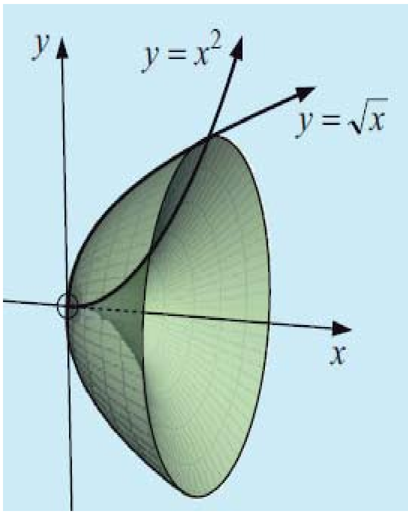
Example 4.81

Find the volume of the solid of revolution generated by revolving the region enclosed by $y = \sqrt{x}$ and $y = x^2$ about the x-axis.

Solution

First, sketch the two functions





Points of intersection are $(0,0)$ and $(1,1)$, then we take the integral between 0 and1. The function $y = \sqrt{x}$ is above the function $y = x^2$.

$$V = \pi\int_0^1 \left[\left(\sqrt{x}\right)^2 - \left(x^2\right)^2\right]dx$$

$$= \pi\int_0^1 \left(x - x^4\right)dx$$

$$= \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 \quad = \pi\left(\frac{1}{2} - \frac{1}{5} - 0 + 0\right) \quad = \frac{3\pi}{10} \text{ cubic units}$$

Example 4.82

Find the volume of the solid of revolution generated when the region enclosed by $y=\sqrt{x}$, $y=2$ and $x=0$ is revolved about the $y-axis$.

Solution

Here, $y=\sqrt{x} \Rightarrow x=y^2$

The volume is

$$V=\pi\int_0^2 \left(y^2\right)^2 dy=\pi\int_0^2 y^4 dy=\pi\left[\frac{y^5}{5}\right]_0^2=\frac{32\pi}{5} \quad cubic\ units$$

Application activity 4.24

1. Find the volume of the solid formed when the region enclosed by the given curves is revolved about the $x-axis$;
   a) $y=x^2,\ x=0,\ x=2,\ y=0$
   b) $y=1+x^3,\ x=1,\ x=2,\ y=0$
   c) $y=9-x^2,\ y=0$
2. Find the volume of the solid formed when the region enclosed by the given curves and the lines is revolved about the $y-axis$;
   a) $y=x^3,\ x=0,\ y=1$
   b) $x=\sqrt{1+y},\ x=0,\ y=3$
   c) $x=\csc y,\ y=\frac{\pi}{4},\ y=\frac{3\pi}{4},\ x=0$

Notice

We can find the volume using an alternative method called **"The shell method"**.

When using this method, it is necessary to integrate perpendicular to the axis of revolution (unlike the disc or washer method). If the region is revolved about a horizontal line, integrate with respect to y, and if the region is revolved about a vertical line, integrate with respect to x.

As always, the radius is the distance to the axis or revolution. For every radius R, it is necessary to find the corresponding height of the shell H .

The values of R and H need to be expressed in terms of the variable of integration.

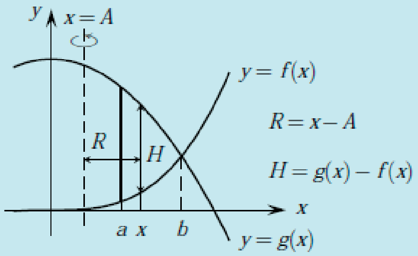


Figure. 4.6. *Region revolved about a vertical line*

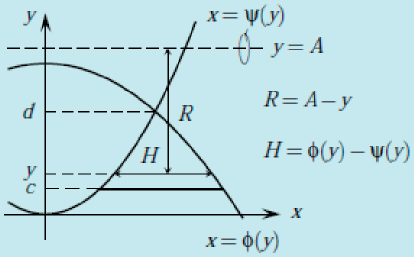


Figure. 4.7. *Region revolved about a horizontal line*

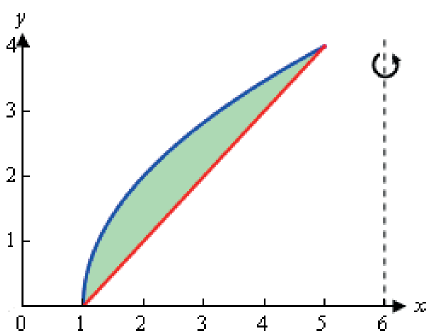
Basing on figure 4.6 and figure 4.7, the volume will equal $V = 2\pi \int_a^b RHdx$ if integrating by x and $V = 2\pi \int_c^d RHdy$ if integrating by y.

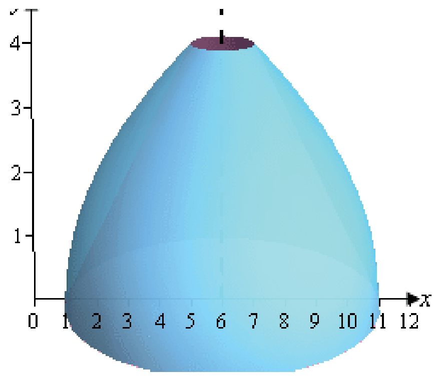
Example 4.83

Determine the volume of the solid obtained by rotating the region bounded by $y = 2\sqrt{x-1}$ and $y = x-1$ about the line $x = 6$.

Solution

Here, a graph of the bound region and solid are:





The figure formed is a typical cylinder. Again, the sketch on the left is here to provide some context for the sketch on the right.

The cross sectional area of the solid is

$$A(x) = 2\pi(radius)(height)$$

$$= 2\pi(6-x)\left(2\sqrt{x-1}-x+1\right)$$

$$= 2\pi\left(12\sqrt{x-1}-6x+6-2x\sqrt{x-1}+x^2-x\right)$$

$$= 2\pi\left(x^2-7x+6+12\sqrt{x-1}-2x\sqrt{x-1}\right)$$

Now, the first cylinder will cut into the solid at x=1 and the final cylinder will cut into the solid at x=5; so they are our limits.

The volume of the solid formed is given by:

$$V = 2\pi\int_1^5\left(x^2-7x+12\sqrt{x-1}-2x\sqrt{x-1}\right)dx$$

$$= 2\pi\left(\frac{x^3}{3}-\frac{7}{2}x^2+6x+8(x-1)^{\frac{3}{2}}-\frac{4}{3}(x-1)^{\frac{3}{2}}-\frac{4}{5}(x-1)^{\frac{5}{2}}\right)\Bigg|_1^5$$

$$= 2\pi\left(\frac{136}{15}\right) = \frac{272\pi}{15}\,cubic\,units$$

Example 4.84

Determine the volume of the solid obtained by rotating the region bounded by $x=(y-2)^2$ and $y=x$ about the line $y=-1$.

Solution

We should first get the points of intersection.

$$(y-2)^2 = y \Leftrightarrow y^2-4y+4=y$$

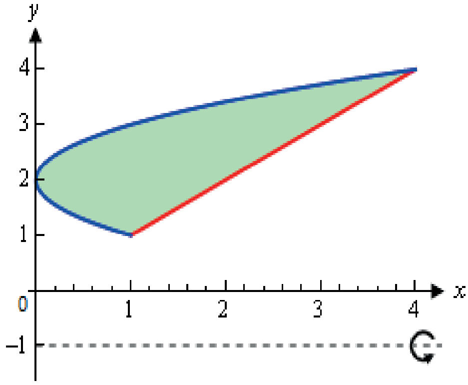
$$\Leftrightarrow y^2-4y+4=y$$

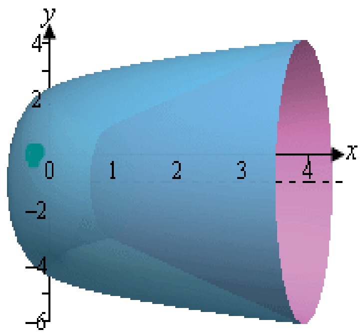
$$\Leftrightarrow y^2-5y+4=0$$

$$\Leftrightarrow (y-4)(y-1)=0$$

So, the two curves will intersect at $y=1$ and $y=4$.

Thus, the sketches of the bound region and the solid are:





The cross sectional area for this cylinder is

$$A(y)=2\pi(radius)(width)\ =2\pi(y+1)\left[y-(y-2)^2\right]$$
$$=2\pi\left(-y^3+4y^2+y-4\right)$$

The first cylinder will cut into the solid at $y=1$ and the final cylinder will cut in at $y=4$.

The volume of the solid formed is given by:

$$V=2\pi\int_1^4\left(-y^3+4y^2+y-4\right)dy=2\pi\left(-\frac{y^4}{4}+\frac{4}{3}y^3+\frac{y^2}{2}-4y\right)\Bigg|_1^4$$
$$=\frac{63\pi}{2}\,cubic\,units$$

Example 4.85

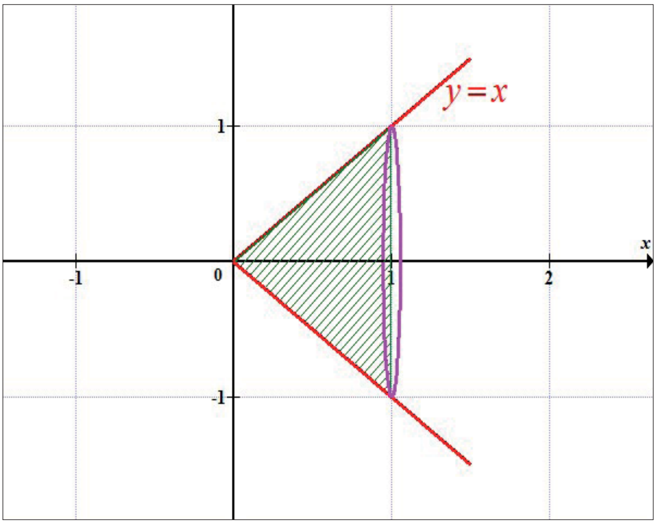
Use the shell method to find the volume of the solid generated when the region bound by the lines $y=0, x=1$ and $y=x$ is revolved about the;

a) $x-axis$ b) line $y=-1$ c) line $y=3$

Solution

In this example, the axis of revolution in each case is a horizontal line, and therefore it will be necessary to integrate by y.

a)

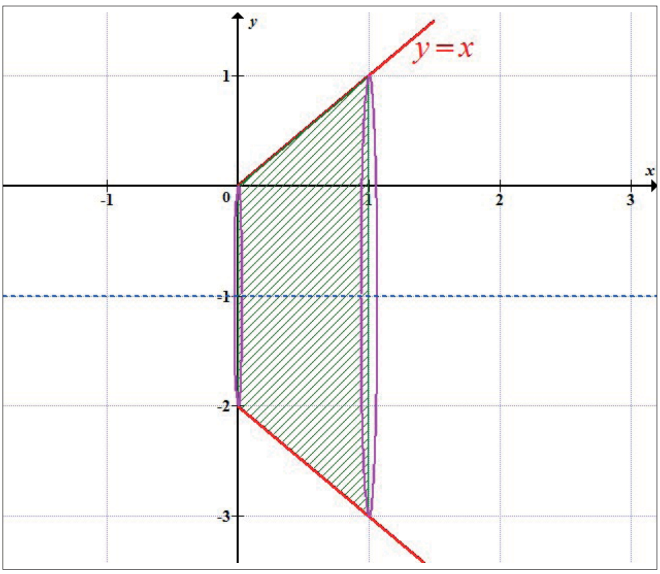


The radius is the distance between the axis of revolution and the current value of $y : R = y - 0 = y$. The height of the shell is the horizontal distance between the line $y = x$ and the line $x = 1$ i.e. $H = 1 - y$.

Therefore, the volume of the solid formed is

$$V = 2\pi \int_c^d RH dy = 2\pi \int_0^1 y(1-y)\,dy = 2\pi \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 = \frac{2\pi}{6} = \frac{\pi}{3}$$

b)



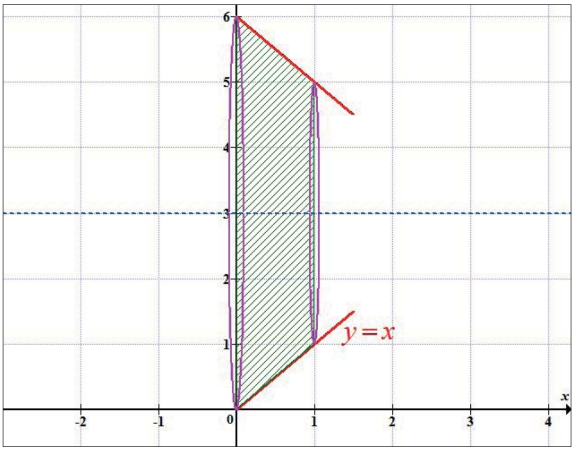
The radius is again the distance between the axis of revolution $y = -1$ and the current value of y $i.e.$ $R = y-(-1) = y+1$.

The height of the shell will be $H = 1-y$, exactly as in a).

Hence, the volume is

$$V = 2\pi\int_0^1 (y+1)(1-y)\,dy = 2\pi\int_0^1 (1-y^2)\,dy = 2\pi\left[y-\frac{y^3}{3}\right]_0^1 = \frac{4\pi}{3}$$

c)



Here, $R = 3-y$ while the height again remains the same. $H = 1-y$

Therefore, the volume of the solid formed is

$$V = 2\pi\int_0^1 (3-y)(1-y)\,dy = 2\pi\int_0^1 (3-4y+y^2)dy = 2\pi\left[3y-2y^2+\frac{y^3}{3}\right]_0^1 = \frac{8\pi}{3}$$

Application activity 4.25

1. Use the shell method to find the volume of the solid generated when the region bound by the lines $x=0, x=1$ and $y=x$ is revolved about the;

   a) line $x=1$ b) $y-axis$

   c) line $x=4$

2. Use cylindrical shells to find the volume of solid generated when the region enclosed between $y=\sqrt{x}$, $x-axis$, and the line $x=4$, is revolved about the $x-axis$.

3. The region bound by the curve $y=\sqrt{x}$, . the lines $x=1,\ x=4$ and the $x-axis$ is revolved about $y-axis$ to generate a solid. Find the volume of the solid by the shell method.

4. Use cylindrical shells to find the volume of the solid generated when the region R in the first quadrant enclosed between $y=x$ and $y=x^2$ is revolved about the $y-axis$.

4.5.3. Calculation of the arc length of a curved line

Activity 4.25

Consider a curve given by the function $y=f(x)=(x-1)^{\frac{3}{2}}$

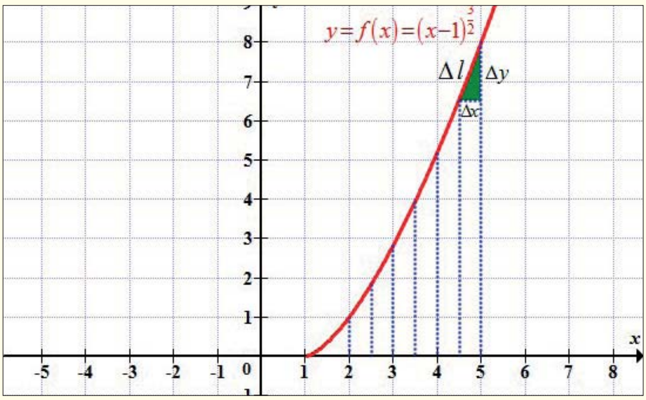


***Figure 4.6.** Arc length of a curved line*

In the triangle shown (shaded region),

1. By Pythagorean Theorem, find Δl.
2. As the step size is made smaller and smaller, $\Delta x \to dx,\ \Delta y \to dy$ and $\Delta l \to dl$.
   From the result obtained in 1), write expression equivalent to dl using $f'(x)$ where $f(x)=(x-1)^{\frac{3}{2}}$.
3. Take definite integral on both sides of the relation obtained in 2), for $2 \le x \le 5$, to find l.

From activity 4.25, we get the arch length of the curve of the function $f(x)$, from $x=a$ to $x=b$ by the formula

$$l = \int_a^b \sqrt{1+\left[f'(x)\right]^2}\,dx$$

Example 4.86

Find the length of a line whose slope is -2 given that line extends from $x=1$ to $x=5$.

Solution

We need the equation of the line in order to find $f'(x)$ but for the purpose of this example, we are given that the slope is -2 and we know that the slope is given by the derivative of the function, then $f'(x)=-2$

Hence,

$$L=\int_1^5 \sqrt{1+\left(f'(x)\right)^2}\,dx=\int_1^5 \sqrt{1+\left(-2\right)^2}\,dx=\int_1^5 \sqrt{5}dx=\sqrt{5}\int_1^5 dx$$

$$=\sqrt{5}\left[x\right]_1^5=\sqrt{5}\left(5-1\right)=4\sqrt{5}\ \textit{units of length}$$

Example 4.87

Find the length of the circle of radius R and centre $\left(0,0\right)$.

Solution

The circle of radius R and centre $\left(0,0\right)$ has equation:

$$x^2+y^2=R^2 \Leftrightarrow y^2=R^2-x^2 \Rightarrow y=\pm\sqrt{R^2-x^2}$$

$$y'=\pm\frac{x}{\sqrt{R^2-x^2}} \Rightarrow \left(y'\right)^2=\frac{x^2}{R^2-x^2}$$

$L=\int_a^b \sqrt{1+(y')^2}\,dx$ But $a=-R$ and $b=R$

$$\Rightarrow L=2\int_{-R}^{R}\left(1+\frac{x^2}{\sqrt{R^2-x^2}}\right)dx$$

We multiply the above result by 2 because we have two parts; one above $x-axis$ and another below $x-axis$.

$$L=2\int_{-R}^{R}\sqrt{\frac{R^2-x^2+x^2}{R^2-x^2}}\,dx=2\int_{-R}^{R}\sqrt{\frac{R^2}{R^2-x^2}}\,dx=2\int_{-R}^{R}\frac{R}{\sqrt{R^2-x^2}}\,dx$$

$$=2\int_{-R}^{R}\frac{Rdx}{\sqrt{R^2+x^2}}=2R\int_{-R}^{R}\frac{dx}{\sqrt{R^2+x^2}}=2R\left[\arcsin\frac{x}{R}\right]_{-R}^{R}$$

$$= 2R\left(\arcsin\frac{R}{R} - \arcsin\frac{-R}{R}\right) = 2R\left(\arcsin 1 - \arcsin -1\right)$$

$$= 2R\left(\frac{\pi}{2} - (-\frac{\pi}{2})\right) = 2R\pi = 2\pi R \ \textit{units of length}$$

Notice

For a curve expressed in the form $x = g(y)$ where g' is continuous on $[c, d]$, the arc length from $y = c$ to $y = d$ is given by $L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$

Example 4.88

Find the arc length of the curve $y = x^{\frac{3}{2}}$ from $(1,1)$ to $(2, 2\sqrt{2})$

Solution

$y = x^{\frac{3}{2}} \Rightarrow x = y^{\frac{2}{3}}$. Hence, $g(y) = y^{\frac{2}{3}}$ and $g'(y) = \frac{2}{3}y^{-\frac{1}{3}}$

The arc length is;

$$L = \int_1^{2\sqrt{2}} \sqrt{1 + \left[\frac{2}{3}y^{-\frac{1}{3}}\right]^2}\, dy \quad = \int_1^{2\sqrt{2}} \sqrt{\frac{9y^{\frac{2}{3}} + 4}{9y^{\frac{2}{3}}}}\, dy$$

$$= \int_1^{2\sqrt{2}} \sqrt{1 + \frac{4}{9}y^{-\frac{2}{3}}}\, dy \quad = \int_1^{2\sqrt{2}} \frac{1}{3y^{\frac{1}{3}}} \sqrt{9y^{\frac{2}{3}} + 4}\, dy$$

$$= \frac{1}{3}\int_1^{2\sqrt{2}} y^{-\frac{1}{3}} \sqrt{9y^{\frac{2}{3}} + 4}\, dy$$

Let $t = 9y^{\frac{2}{3}} + 4 \Rightarrow dt = 6y^{-\frac{1}{3}}dy \Rightarrow dy = \dfrac{dt}{6y^{-\frac{1}{3}}}$

If $y = 1,\ t = 12$ and if $y = 2\sqrt{2},\ t = 22$

$$L = \frac{1}{3}\int_{13}^{22} y^{-\frac{1}{3}} \sqrt{t}\, \frac{dt}{6y^{-\frac{1}{3}}} = \frac{1}{18}\int_{13}^{22} \sqrt{t}\, dt = \frac{1}{18}\int_{13}^{22} t^{\frac{1}{2}}\, dt = \frac{1}{18}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_{13}^{22}$$

$$=\frac{1}{18}\times\frac{2}{3}\left[t^{\frac{3}{2}}\right]_{13}^{22}=\frac{1}{27}\left[(22)^{\frac{2}{3}}-(13)^{\frac{2}{3}}\right]=\frac{22\sqrt{22}-13\sqrt{13}}{27}\ \textit{units of length}$$

Application activity 4.26

1. Find the arc length of the curve $y=3x^{\frac{3}{2}}-1$ from $x=0$ to $x=1$.
2. Find the arc length of the curve $y=x^{\frac{2}{3}}$ from $x=1$ to $x=8$.
3. Determine the arc length of the curve $y=\ln(\sec x)$ from $x=0$ to $x=\frac{\pi}{4}$.
4. Find the arc length of the curve $x=\frac{2}{3}(y-1)^{\frac{3}{2}}$ for $1\le y\le 4$.

Notice

Further applications in physics

1. Work done

Work is defined as the amount of energy required to perform a physical task. When force is constant, work can simply be calculated using the equation $W=F\cdot d$ where W is work, F is a constant force, and d is the distance through which the force acts. The units of work are commonly $Newton-metres(Nm)$, $Joules(J)$, $Foot-pound(ft-lb)$. Frequently, the force is not constant and will change over time. In order to obtain the amount of work done with a variable force, the following integral equation must be used $W=\int_a^b f(x)\,dx$

where W is work, $f(x)$ is force as a function of distance, and x is distance.

Remark

In physics, the kinetic energy of an object is the energy which it possesses due to its motion. It is defined as the work needed to accelerate a body of a given mass from rest to its stated velocity.

Having gained this energy during its acceleration, the body maintains this kinetic energy unless its speed changes. The same amount of work is done by the body in decelerating from its current speed to a state of rest.

Example 4.89

A spring has a natural length of $1\,meter$. A force of $25\,N$ stretches the string by $\frac{1}{4}$ of a metre. Determine how much work is done by stretching the spring

a) $2\,meters$ beyond its natural length

b) from a length of $1.5\,meters$ to $2.5\,meters$

Solution

We first determine the string constant, k. Because the force is 25 N when $x=\frac{1}{4}m=0.25m$, we can use Hooke's law to determine k.

$$f(x)=kx$$

$$25N=k\left(\frac{1}{4}m\right)\Rightarrow k=100\frac{N}{m}$$

And $f(x)=100x$

Hence,

a) $W=\int_0^2 100x\,dx=\left[\frac{100}{2}x^2\right]_0^2$

$$=50(4-0)$$

$$=200Nm \quad or \quad 200J$$

b) Here, we need to pay attention to boundaries. If the spring is not stretched, no matter what its length is, the lower boundary must be zero. If the string is stretched to a certain length, then we need to subtract the natural length from that value to obtain the lower limit of integration. So, here

$$W=\int_{0.5}^{1.5}100x\,dx=\left[\frac{100}{2}x^2\right]_{0.5}^{1.5}=50\left((1.5)^2-(0.5)^2\right)=100\,Nm \quad or \quad 100\,J$$

Example 4.90

A force of 40 N is required to hold a spring that has been stretched from 10 cm to 15 cm.

How much work is done in stretching the spring from 15 cm to 18 cm?

Solution

According to Hooke's law, the force required to hold the spring stretched x metres beyond its natural length is $f(x)=kx$. When the spring is stretched from $10\,cm$ to $15\,cm$, the amount by which it has been stretched is $5cm = 0.05m$. This means that $f(0.05)=40$,

so, $0.05k = 40, \quad k = \dfrac{40}{0.05} = 800$

Thus, $f(x)=800x$ and the work done in stretching the spring from $15\,cm$ to $18\,cm$ is

$$W = \int_{0.05}^{0.08} 800x\,dx = \left[800\frac{x^2}{2}\right]_{0.05}^{0.08} = 400\left[(0.08)^2 - (0.05)^2\right] = 1.56\,J$$

2. Motion problems

Recall that for some displacement function $s(t)$,

the velocity is given by $v(t) = \dfrac{d[s(t)]}{dt}$ and

the acceleration is $a(t) = \dfrac{d^2[s(t)]}{dt^2} = \dfrac{d[v(t)]}{dt}$.

So, given a velocity function, we can determine the displacement function by the integration $s(t) = \int v(t)\,dt$.
Using the displacement function, we can determine the displacement in a time interval $a \le t \le b$.
Thus, $S = \int_a^b v(t)\,dt$

Also, given an acceleration function, we can determine the velocity function by the integration $v(t) = \int a(t)\,dt$.
Using the velocity function, we can determine the velocity in a time interval $a \le t \le b$; $V = \int_a^b a(t)\,dt$

Example 4.91

The velocity of a body t seconds after a certain instant is given by $v(t) = (2t^2 + 5)\ ms^{-1}$. How far does it travel in the first 4 seconds of motion?

Solution

$$S=\int_0^4\left(2t^2+5\right)dt=\left[\frac{2t^3}{3}+5t\right]_0^4=\left(\frac{2\left(4^3\right)}{3}+5(4)\right)-0=\frac{188}{3}\ m$$

Thus, the distance travelled is $\frac{188}{3}\ m$

Example 4.92

An object starts from rest and has an acceleration of $a(t)=t^2$. What is its;

a) velocity after 3 seconds?

b) position after 3 seconds?

Solution

We will take the initial position of our object to be the origin of our coordinate system. Thus, $s(0)=0$. Since the object started at rest, we have $v(0)=0$. This data will be useful in the determination of the constants in the integrations.

a) $v(t)=\int a(t)\,dt=\int t^2\,dt=\frac{1}{3}t^3+c$

$v(0)=\frac{1}{3}(0)^3+c=0\Rightarrow c=0$

Then, $v(t)=\frac{1}{3}t^3$

After 3 seconds, we have

$v(3)=\frac{1}{3}(3)^3=9$ units of velocity.

b) $s(t)=\int v(t)\,dt=\int\frac{1}{3}t^3\,dxt=\frac{1}{12}t^4+c$

$s(0)=\frac{1}{12}(0)^4+c=0\Rightarrow c=0$

Then, $s(t)=\frac{1}{12}t^4$

After 3 seconds, we have

$s(3)=\frac{1}{12}(3)^4=\frac{27}{4}$ units of length.

Unit summary

1. Differentials

The exact change Δy in y is given by $\Delta y = f(x+\Delta x) - f(x)$.

But if the change Δx is small, then we can get a good approximation to Δy by using the fact that $\frac{\Delta y}{\Delta x}$ is approximately the derivative $\frac{dy}{dx}$. Thus,

$$\Delta y = \frac{\Delta y}{\Delta x}\Delta x \approx \frac{dy}{dx}\Delta x = f'(x)\Delta x$$

If we denote the change of x by dx instead of Δx,

then the change Δy in y is approximated by the differential dy, that is, $\Delta y \approx dy = f'(x)dx$

Whenever one makes an approximation, it is wise to try and estimate how big the error might be.

Relative change in x is $\frac{\Delta x}{x}$

Percentage change in x is $100 \times \frac{\Delta x}{x}$

2. Indefinite integrals

Integration can be defined as the inverse process of differentiation.

If $y = f(x)$ then

$$\frac{dy}{dx} = f'(x) \Leftrightarrow \int \frac{dy}{dx}dx = f(x) + c$$

Or equivalently

$$\int \frac{dy}{dx}dx = y + c$$

This is called **indefinite integration** and c is the constant of integration.

3. Basic integration formula

Exponential functions

a) $\int x^n dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$

b) $\int e^x dx = e^x + c$

c) $\int a^x dx = \frac{a^x}{\ln a} + c$

Rational functions

a) $\int \frac{1}{x}dx = \ln|x| + c$ b) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + c$

c) $-\int \frac{dx}{a^2+x^2} = \frac{1}{a}\operatorname{arc\,cot}\frac{x}{a} + c$

d) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

e) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + c$

Irrational functions

a) $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + c$

b) $-\int \frac{dx}{\sqrt{a^2-x^2}} = \arccos\frac{x}{a} + c$

c) $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|\frac{x+\sqrt{x^2+a^2}}{a}\right| + c$

d) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| + c$

e) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\operatorname{arc\,sec}\frac{x}{a} + c$

f) $-\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\operatorname{arc\,csc}\frac{x}{a} + c$

Trigonometric functions

a) $\int \sin x\,dx = -\cos x + c$ b) $\int \cos x\,dx = \sin x + c$

c) $\int \sec^2 x\,dx = \tan x + c$ d) $\int \csc^2 x\,dx = -\cot x + c$

e) $\int \tan x\,dx = -\ln|\cos x| + c$ f) $\int \cot x\,dx = \ln|\sin x| + c$

g) $\int \sec x\,dx = \ln|\sec x + \tan x| + c$

h) $\int \csc x\,dx = -\ln|\csc x + \cot x| + c$

i) $\int \sec x \tan x\,dx = \sec x + c$

j) $\int \csc x \cot x\,dx = -\csc x + c$

4. Non basic integration

I. Integration by substitution

In evaluating $\int f(x)dx$ when $f(x)$ is not a basic function:

if $f(x)=g'(x)g(x)$ or $f(x)=\dfrac{g'(x)}{g(x)}$ or

$f(x)=h(g(x))g'(x)$, you let $u=g(x)$.

II. Integration by parts

To integrate a product of functions, try the formula for integration by parts
$\int u\dfrac{dv}{dx}dx=uv-\int v\dfrac{du}{dx}dx$ or $\int udv=uv-\int vdu$

An effective strategy is to choose for $\dfrac{dv}{dx}$ the most complicated factor that can readily be integrated. Then we differentiate the other part, u, to find $\dfrac{du}{dx}$.

The following table can be used:

u	v'
Logarithmic function	Polynomial function
Polynomial function	Exponential function
Polynomial function	Trigonometric function
Exponential function	Trigonometric function
Trigonometric function	Exponential function
Inverse trigonometric function	Polynomial function

Applying the method of integration by parts, the power of integrand is reduced and the process is continued till we get a power whose integral is known or which can be easily integrated. This process is called **Reduction formula**.

III. Integration by partial fractions

Remember that:

A **rational function** is a function of the form

$f(x)=\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials.

A **proper rational function** is a rational function in which the degree of $P(x)$ is strictly less than the degree of $Q(x)$.

The problem of integrating rational functions is really the problem of integrating *proper* rational functions since **improper rational functions** (i.e. those in which the degree of $P(x)$ is greater than or equal to the degree of $Q(x)$) and can always be rewritten as the sum of a polynomial and a proper rational function.

The integrals of proper rational functions are found by **partial fraction expansion** of the integrand into simple fractions.

There are 4 types of simple fractions:

a) Fractions of the type $\dfrac{A}{x-a}$

The integrals of such fractions are easily found:

$$\int \frac{A}{x-a}dx = A\ln|x-a|+c$$

b) Fractions of the type $\dfrac{A}{(x-a)^n}$, where n is a natural number greater than 1.

The integrals of such fractions are easily found:

$$\int \frac{A}{(x-a)^n}dx = A\int (x-a)^{-n}\,dx = \frac{A}{1-n}(x-a)^{1-n}+c$$

c) Fractions of the type $\dfrac{Ax+B}{x^2+px+q}$, where $p^2-4q<0$

The integrals of such fractions are found by completing the square in the denominator and subsequent substitution which leads to rational integrals of the form $\int \dfrac{du}{u^2+k^2}$ *or* $\int \dfrac{du}{u^2-k^2}$ *or* $\int \dfrac{du}{k^2+u^2}$.

d) Fractions of the type $\dfrac{Ax+B}{(x^2+px+q)^n}$,

where $p^2-4q<0$ and n is a natural number greater than 1.

Integration of this type of fraction will not be considered in this course.

Expansion of proper rational functions in partial fractions is achieved by first factoring the denominator and then writing the type of partial fraction (with unknown coefficients in the numerator) that corresponds to each term in the denominator:

(i) if the denominator contains $(x-a)$, then the partial fraction expansion will contain $\frac{A}{x-a}$.

(ii) if the denominator contains $(x-a)^n$, then the partial fraction expansion will contain

$$\frac{A}{(x-a)^n}+\frac{B}{(x-a)^{n-1}}+\frac{C}{(x-a)^{n-2}}=\ldots+\frac{Z}{(x-a)}.$$

(iii) if the denominator contains $\left(x^2+px+q\right)$ where $p^2-4q<0$, then the partial fraction expansion will contain $\frac{Ax+B}{x^2+px+q}$.

The unknown coefficients (A, B, etc.) are then found by one of the two ways:

by inserting concrete values of the variable or by using the method of undetermined coefficients.

5. Integration of irrational functions

- **Integrals of the form** $\int\frac{dx}{\sqrt{ax^2+bx+c}}$

The integrals of such fractions are found by completing the square in the denominator and subsequent substitution which lead to irrational integrals of the form

$$\int\frac{du}{\sqrt{u^2+k^2}}\ or\int\frac{du}{\sqrt{u^2-k^2}}\ or\ \int\frac{du}{\sqrt{k^2+u^2}}$$

- **Integrals of the form** $\int\frac{px+q}{\sqrt{ax^2+bx+c}}dx$

The numerator is written as the sum of two parts. One part is the derivative of radicand and the other part is a constant only, i.e.

$$\int\frac{px+q}{\sqrt{ax^2+bx+c}}dx=k_1\int\frac{\frac{d}{dx}\left(ax^2+bx+c\right)}{\sqrt{ax^2+bx+c}}dx+k_2\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

- **Integrals of the form** $\int \frac{px^2+qx+r}{\sqrt{ax^2+bx+c}}dx$

  The numerator is written as the sum of three parts. One part is the same as radicand, the second part is derivative of radicand and the last part is a constant only, i.e.

$$\int \frac{px^2+qx+r}{\sqrt{ax^2+bx+c}}dx \qquad ax \quad bx \quad c$$

$$= k_1\int \frac{ax^2+bx+c}{\sqrt{ax^2+bx+c}}dx + k_2\int \frac{\frac{d}{dx}\left(ax^2+bx+c\right)}{\sqrt{ax^2+bx+c}}dx + k_3\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

6. Integration of trigonometric functions

- **Integrals of the form** $\int \frac{dx}{a\sin x + b\cos x + c}$

  You can use t-formulae by letting $t = \tan\frac{x}{2}$.

- **Integrals of the form** $\int \frac{dx}{a+b\cos^2 x}$ **or** $\int \frac{dx}{a+b\sin^2 x}$

  Here also you can use t-formulae.

  In integrating the trigonometric functions containing product or power, transforming product or power into sum (or difference) leads to basic integration.

7. Definite integration

Remember that integrals containing an arbitrary constant c in their results are called **indefinite integrals** since their precise value cannot be determined without further information

a) Definite integrals are those in which limits are applied.

If an expression is written as $\left[F(x)\right]_a^b$, 'b' is called the upper limit and 'a' the lower limit.

The operation of applying the limits is defined as:

$$\left[F(x)\right]_a^b = F(b) - F(a)$$

For example, the increase in the value of the integral $f(x)$ as x increases from 1 to 3 is written

as $\int_1^3 f(x)dx$.

The **definite integral** from $x = a$ to $x = b$ is defined as the area under the curve between

those two values. This is written as $\int_a^b f(x)dx$

b) The **mean value** of a function $y = f(x)$ over the range $]a,b[$ is the value the functions would have if it were constant over the range but with the same area under the graph. The mean value of $y = f(x)$ over the range $]a,b[$ is $\overline{f(x)} = \frac{1}{b-a}\int_a^b f(x)dx$.

c) The **root mean square** value (R.M.S. value) is the square root of the mean value of the square of y. The r.m.s. value from $x = a$ to $x = b$ is given by;

$$R.M.S. = \sqrt{\frac{\int_a^b f^2(x)dx}{b-a}}$$

8. Improper integral

The definite integral $\int_a^b f(x)dx$ is called an **improper integral** if one of two situations occurs:

- The limit a or b (or both bounds) are infinites.
- The function $f(x)$ has one or more points of discontinuity in the interval $[a,b]$.

Let $f(x)$ be a continuous function on the interval $[a,+\infty[$ or $]-\infty,b]$.

We define the improper integral as $\int_a^{+\infty} f(x)dx = \lim_{n\to+\infty}\int_a^n f(x)dx$

Or $\int_{-\infty}^b f(x)dx = \lim_{n\to-\infty}\int_n^b f(x)dx$ respectively.

If these limits exist and are finite, then, we say that the improper integrals are **convergent** otherwise the integrals are **divergent.**

Let $f(x)$ be a continuous function for all real numbers. By Chasles theorem

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx$$

If for real number, c, both integrals on the right side are convergent, then we say that the integral $\int_{-\infty}^{+\infty} f(x)dx$ is also **convergent**; otherwise it is **divergent.**

9. Applications

Integration has many applications, some of which are listed below:

a) The **area** between two functions $f(x)$ and $g(x)$ where $f(x) \le g(x)$ in $[a,b]$ is given by

$$\int_{a}^{b} [g(x) - f(x)]dx = \int_{a}^{b} g(x)dx - \int_{a}^{b} f(x)dx$$

b) Volume

The volume of a solid of revolution can be found using one of the following methods:

the disc, washer method and the shell method.

In any of the methods, when finding volume, it is necessary to integrate along the axis of revolution; if the region is revolved about a horizontal line, integrate with respect to x, and if the region is revolved about a vertical line, integrate with respect to y.

(i) Disc method

The volume of the solid of revolution bound by the curve $f(x)$ about the $x-axis$ calculated from $x=a$ to $x=b$, is given by $\pi\int_{a}^{b} y^2 dx$.

Volume of the solid generated by revolution of the area bound by the curve $y=f(x)$ about the

$y-axis$ is given by $\pi\int_{a}^{b} x^2 dy$.

If the axis of revolution is the line parallel to $x-axis$ (say $y=k$), the volume will be

$$\pi\int_{a}^{b} (y-k)^2 dx.$$

(ii) Washer method

If the region bound by outer radius $y_U = g(x)$ (on top) and inner radius $y_L = f(x)$ and the lines $x = a$,

$x = b$ is revolved about $x-axis$, then the volume of revolution is given by: $V = \pi\int_a^b \left([g(x)]^2 - [f(x)]^2\right)dx$

(iii) Shell method

The volume of the solid generated by revolving the region between the curve $x-axis, y = f(x) \geq 0, L \leq a \leq x \leq b$, about a vertical line $x = L$ is

$$V = 2\pi\int_a^b \begin{pmatrix} shell \\ radius \end{pmatrix}\begin{pmatrix} shell \\ height \end{pmatrix}dx$$

HINT for shell method:

Regardness of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are the following:

» Draw the region and sketch a line segment across it, parallel to the axis of revolution. Label the segment's height or length (shell height) and distance from the axis of revolution (shell radius).

» Find the limits of integration for the thickness variable.

» Integrate the product $2\pi\begin{pmatrix} shell \\ radius \end{pmatrix}\begin{pmatrix} shell \\ height \end{pmatrix}$ with respect to the thickness variable $(x\, or\, y)$ to find the volume.

» Length of arc of the curve $y = f(x)$ between the points whose absissas are a and b is $s = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

» The work done by a variable force $F(x)$ in the direction of motion along the $x-axis$ over the interval $[a,b]$ is $W = \int_a^b F(x)\,dx$.

10. Hook's law says that the force required to hold a stretched or compressed spring x units beyond its equilibrium position pulls back with a force $F(x) = kx$ where k is constant called **spring constant**(or **force constant**).

End of unit assessment

1. Find indefinite integrals for question a-l.

a) $\int \frac{x^3}{x^2-9x+20}dx$

b) $\int \frac{3x^3+11x^2+x+3}{x+3}dx$

c) $\int \frac{x^2+10x+24}{(x-2)^2(x+1)}dx$

d) $\int \frac{2x}{(x^2+1)(x-1)}dx$

e) $\int \frac{x^2+4}{(x^2+2x+2)(x+1)}dx$

f) $\int \frac{4x^2+2}{(x^2+1)(x+1)^2}dx$

g) $\int x\cos x dx$

h) $\int 5xe^{4x}dx$

i) $\int x\ln x dx$

j) $\int x^2\sin 3x dx$

k) $\int e^{ax}\cos bx dx$

l) $\int e^{ax}\sin bx dx$

2. Evaluate;

a) $\int_{-1}^{1}(2x^2-x^3)dx$

b) $\int_{-3}^{1}\left(\frac{1}{x^2}-\frac{1}{x^3}\right)dx$

c) $\int_{-6}^{10}\frac{dx}{x+2}$

d) $\int_{-2}^{2}\frac{dx}{x^2+4}$

e) $\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\sin x dx$

f) $\int_{1}^{e}\ln x dx$

3. Given that $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\cos nx\, dx$ and $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin nx\, dx$ where n is a positive integer. Show by using integration by parts that $a_n=0$ and $b_n=-\frac{2}{n}\cos n\pi$.

4. Determine the area enclosed between the curves $y=x^2+1$ and $y=7-x$.

5. Evaluate by integration the area bound by the three straight lines $y=4-x, y=3x$ and $3y=x$.

6. Find the area of the region bound by the curves $y^2 = 2x+1$ and $x-y-1=0$.

7. Determine the volume of the solid generated when the region bound by the lines $y=3x, y=2, y=4$ and $x=0$ is revolved about the $y-axis$.

8. Determine the volume of the solid generated when the region bound by the curve $y^2 = 4x$ and line $y=x$ is revolved about the $x-axis$.

9. Find the volume of the solid generated when the region bound by the curve $y^2 = 4x$ and the line $y=x$ is revolved about the line $x=-1$.

10. Find the volume of the solid generated when the region bound by the curves $y = x - x^2$ and the line $y=0$ is revolved about the:

    a) $the\, x-axis$ b) $the\, y-axis$

    c) line $x=2$ d) line $x=-2$

    e) line $y=-1$ f) line $y=2$.

11. Find the volume of the solid generated when the region in the first quadrant bound by the curve $y^2 - x + 1 = 0$ and the lines x=2 , y=0 is revolved about

    a) $the\, x-axis$ b) $the\, y-axis$

12. In reaction between ethylene bromide and potassium iodide in 99% methanol, $C_2H_4Br_2 + 3KI \to C_2H_4 + 2KBr + KI_3$ it is found that the amount of iodine $I_2, x\, mol\, dm^{-3}$ is related to the time, t minutes, after the reaction began by: $kt = \int_0^x \frac{dc}{(a-c)(a-3c)}$

    where $k = 0.3dm^3\, mol^{-1} \min^{-1}$ is the reaction rate constant and a is the initial concentration of the chemicals.

    a) Evaluate the integral to form t as a function of x.
    b) By rearranging the formula in a) write x in terms of t.
    c) Find the value of x when t becomes very large.

13. The number of atoms, N, remaining in a mass of material during radioactive decay after time t seconds is given by: $N = N_o e^{-\lambda t}$, where N_o and λ are constants. Determine the mean number of atoms in the mass of material for the time period $t=0$ and $t = \frac{1}{\lambda}$.

14. The average value of a complex voltage wave form is given by: $V_{AV} = \frac{1}{\pi}\int_0^{\pi}\left(10\sin\omega t + 3\sin 3\omega t + 2\sin 5\omega t\right)dt$.

    Evaluate V_{AV} correct to 2 decimal places.

15. Find the work required to compress a spring from its equilibrium length of $0.3m$ to $0.2m$ if the force constant is $k = 234\,\mathrm{N/m}$.

16. A spring exerts a force of a ton when stretched 5m beyond its natural length. How much work is required to stretch the spring 6m beyond its natural length?

17. A spring has a natural length of $1m$. A force of $24N$ holds the spring stretched to a total length of $1.8m$.

    a) Find the force constant.

    b) How much work will it take to stretch the spring to 2m beyond its natural length?

    c) How far will a $45N$ force stretch the spring?

18. A swimming pool is built in the shape of rectangular parallelepiped 10m deep, 15m wide, and 20m long.

    a) If the pool is filled 1m below the top, how much work will be required to pump all the water into a drain at the top edge of the pool?

    b) If a one horsepower motor can do 550m of work per second, what size motor is required to empty the pool in one hour?

Unit 5

Differential Equations

Introductory activity

A quantity $y = y(t)$ is said to have an exponential growth model if it increases at a rate that is proportional to the amount of the quantity present, and it is said to have an exponential decay model if it decreases at a rate that is proportional to the amount of the quantity present.

Thus, for an exponential growth model, the quantity $y = y(t)$ satisfies an equation of the form $\frac{dy}{dt} = k.y(t)$ (k is a non-negative constant called annual growth rate). Given that $\frac{dy}{dt} = k.y(t)$ can be written as $\frac{dy}{y} = k.dt$, solve this equation and apply the answer $y = y(t)$ obtained in the following problem:

The size of the resident Rwandan population in 2018 is estimated to 12,089,721 with a growth rate of about 2.37% comparatively to year 2017 (www.statistics.gov.rw/publication/demographic-dividend).

Assuming an exponential growth model and constant growth rate,

1. Estimate the national population at the beginning of the year 2020 and 2030
2. Does this population continue to increasing or to decrease?
3. What are pieces of advice would you provide to policy makers?

A **differential equation** is an equation that involves a function and its derivatives. We can also say, a differential equation makes a statement connecting the value of a quantity to the rate at which that quantity is changing. Differential equations can describe exponential growth and decay, the population growth of species or the change in investment return over time.

Objectives

By the end of this unit, I will be able to:

- Extend the concepts of differentiation and integration to ordinary differential equations.
- State the order and the degree of an ordinary differential equation.
- Express the auxiliary quadratic equation of a homogeneous linear differential equation of second order with constant coefficients.
- Predict the form of the particular solution of an ordinary linear differential equation of second order.

5.1. Definition and classification

Activity 5.1

In each of the following cases, form another equation by eliminating arbitrary constants. Also, write down the order of the highest derivative that is obtained in the equation.

1. $y = Ax + A^2$
2. $y = A\cos x + B\sin x$
3. $y^2 = Ax^2 + Bx + C$

An equation involving a differential coefficient i.e. $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{dr}{dt}$ e.t.c. is called a **differential equation.**

Order of the highest derivative of function that appears in a differential equation is said to be the **order of differential equation**.

First order differential equation; contains only first derivatives apart from dependent variable.

Second order differential equation; contains second derivatives (and with maybe first derivatives).

Degree of a differential equation refers to the **highest power** of the **highest derivative** which occurs in the differential equation.

Differential equations are classified according to the highest derivative which occurs in them.

Consider the following equations:

1. $\left(\frac{dy}{dx}\right)^2 = e^x + 1$ 2. $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E\sin\omega t$

3. $\frac{d^2y}{dx^2} + \sin x\left(\frac{dy}{dx}\right)^3 + 8y = \tan x$ 4. $\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$

Equation 1) above is a first order differential equation (the highest derivative appearing is the first derivative) and degree 2 (the power of the highest derivative is 2).

Equation 2) is a second order differential equation (the highest derivative appearing is the second derivative) and degree 1 (the power of the highest derivative is 1).

Equation 3) is a second order differential equation (the highest derivative appearing is the second derivative) and degree 1 (the power of the highest derivative is 1).

Equation 4) is a second order differential equation (the highest derivative appearing is the second derivative) and degree 2 (the power of the highest derivative is 2).

By a **solution** of differential equation, we mean a continuous function $y(t)$ or $y(x)$ that satisfies the differential equation.

The solution to a given differential equation is obtained by **integration**.

Given a function with arbitrary constants, you form differential equation by eliminating its arbitrary constants using differentiation process.

A differential equation in which there is only one independent variable, so that all the derivatives occurring in it are ordinary derivatives is said to be an ordinary differential equation. The general ordinary differential equation of the

n^{th} order is $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots\ldots, \frac{d^ny}{dx^n}\right) = 0$, or

$F\left(x, y, y', y'', \ldots\ldots, y^{(n)}\right) = 0$.

Example 5.1

$\frac{dy}{dt} = 3y^2\sin(t + y)$ is a differential equation of order one.

Example 5.2

$\frac{d^2y}{dt^2}+y=\cos 2t$ is a differential equation of order two.

Example 5.3

$\frac{d^3y}{dx^3}=e^{-y}+x+\frac{d^2y}{dx^2}$ is a differential equation of order three.

Example 5.4

Given the equation $y=Ax^2$. Find the differential equation and hence state its order.

Solution

$y=Ax^2 \Rightarrow A=\frac{y}{x^2}$

Now, $\frac{dy}{dx}=2Ax$

Since $A=\frac{y}{x^2} \Rightarrow \frac{dy}{dx} \Rightarrow \frac{2y}{x}$ as the required differential equation,

the equation is a first order differential equation.

Application activity 5.1

1. Obtain the differential equation for which the given function is a solution;

   a) $y^2=4a(x+a)$ b) $Ax^2+By^2=1$

   c) $y=ae^{3x}+be^x$ d) $y=e^x(A\cos x+B\sin x)$

   e) $y=a\cos(x+3)$

2. State the order and degree of the following differential equations:

a) $\frac{d^2y}{dx^2}-3x=coy-2\sin x$ b) $\left(\frac{dy}{dx}\right)^4-3x=\cos y-2\sin x$

c) $(y'')^3+(y')^5-2y=x^4$ d) $y\frac{d^2y}{dx^2}+\cos x=0$

e) $\dfrac{d^2y}{dx^2}-9y=0$

f) $\dfrac{dy}{dx}\dfrac{d^3y}{dx^3}+2y^2=1$

g) $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+3y=3x+2$

h) $x^2\left(\dfrac{d^2y}{dx^2}\right)^3+y\left(\dfrac{dy}{dx}\right)^4+y^4=0$

5.2. First order differential equations

5.2.1. Differential equations with separable variables

Activity 5.2

Express each of the following equations in the form $f(y)dy=g(x)dx$ and integrate both sides.

1. $\dfrac{dy}{dx}=\dfrac{x}{y}$

2. $\dfrac{dy}{dx}=x^2y^3$

A general differential equation of the 1st order can be written in the form;

$F\left(x,y,\dfrac{dy}{dx}\right)=0$ or $\dfrac{dy}{dx}=f(x,y)$

The simplest is that in which the variables are separable: $\dfrac{dy}{dx}=g(x)h(y)$

To solve the differential equation, we write it in the separated form; $\dfrac{dy}{h(y)}=g(x)dx$ and integrate both sides $\displaystyle\int\frac{dy}{h(y)}=\int g(x)dx$.

Example 5.5

Solve $\dfrac{dy}{dx}=\dfrac{x^2+1}{4}$

Solution

$$\frac{dy}{dx}=\frac{x^2+1}{4}\Leftrightarrow 4dy=\left(x^2+1\right)dx$$

$$\int 4dy = \int (x^2+1)\,dx$$

$$\Rightarrow 4y = \frac{x^3}{3} + 2x + c \Rightarrow y = \frac{x^3}{12} + \frac{x}{2} + c;\ \textit{general solution}$$

Example 5.6

Solve the differential equation, $\dfrac{dy}{dx} = \dfrac{y}{x-6}$

Solution

$$\frac{dy}{dx} = \frac{y}{x-6} \Leftrightarrow \frac{dy}{y} = \frac{dx}{x-6}$$

$$\int \frac{dy}{y} = \int \frac{dx}{x-6}$$

$$\Rightarrow \ln|y| = \ln|x-6| + c$$

$$\Rightarrow \ln|y| = \ln|x-6| + \ln k,\quad \ln k = c$$

$$\Rightarrow \ln|y| = \ln|k(x-6)|$$

$\Rightarrow y = k(x-6)$, k is a constant; general solution.

Application activity 5.2

Solve the following differential equations:

1. $\dfrac{dy}{dx} = \dfrac{y}{2x}$

2. $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$

3. $\dfrac{dy}{dx} = 1 + y^2$

4. $(x+1)\dfrac{dy}{dx} = x(y^2+1)$

5.2.2. Simple homogeneous equations

Activity 5.3

1. Given the relation $f(x,y)=x^2+xy$, replace x with tx and y with ty and re-write the given relation in the form $t^n f(x,y)$. Deduce the value of n.

2. Consider the differential equation $\frac{dy}{dx}=f(x,y)$. By letting $z=\frac{y}{x}$, write equivalent relation to the given equation in function of x and z.

3. Suppose that $f(tx,ty)=f(x,y)$, by letting $t=\frac{1}{x}$ and using relation obtained in 2), write a new relation such that the variables x and z are separated.

A function $f(x,y)$ is called **homogeneous** of degree n if

$f(tx,ty)=t^n f(x,y)$ for all suitably restricted x,y and t.

This means that if x and y are replaced with tx and ty, t^n factors out of the resulting function.

Example 5.7

Show that $\sqrt{x^2+y^2}$ is homogeneous of degree 1.

Solution

$$\sqrt{(tx)^2+(ty)^2}=\sqrt{t^2(x^2+y^2)}$$

$=t^1\sqrt{x^2+y^2}$; where n=1 as required

Example 5.8

Show that $\sin\frac{x}{y}$ is homogeneous of degree 0.

Solution

$$\sin\frac{tx}{ty}=\sin t\times t^{-1}\frac{x}{y}\ =\sin t^{0}\frac{x}{y}=t^{0}\sin\frac{x}{y}$$

Since t^0 then the degree is 0.

Notice

The differential equation; $M(x,y)dx+N(x,y)dy=0$ is said to be homogeneous if M and N are homogeneous functions of the same degree.

This equation can then be written in the form $\dfrac{dy}{dx}=f(x,y)$

where $f(x,y)=\dfrac{-M(x,y)}{N(x,y)}$ is clearly homogeneous of degree 0.

We solve this equation by letting $z=\dfrac{y}{x}$. Which reduces the equation to variable separable.

Example 5.9

Solve $(x+y)dx-(x-y)dy=0$

Solution

We write the equation in the form $\dfrac{dy}{dx}=f(x,y)$

$$\frac{dy}{dx}=\frac{x+y}{x-y}$$

Since, this equation is homogeneous of degree 0, we know that it can be expressed as a function of $z=\dfrac{y}{x}$, this comes by dividing the

numerator and the denominator by x; $\dfrac{dy}{dx}=\dfrac{1+\frac{y}{x}}{1-\frac{y}{x}}$

But $y=zx\ \Rightarrow\dfrac{dy}{dx}=z+x\dfrac{dz}{dx}$

Separating the variables gives:

$$\frac{1-z}{1+z^2}dz=\frac{dx}{x}\Rightarrow\left(\frac{1}{1+z^2}-\frac{z}{1+z^2}\right)dz=\frac{dx}{x}\ \Rightarrow\frac{1}{1+z^2}\ dz-\frac{z}{1+z^2}\ dz=\frac{dx}{x}$$

On integration, we get $\arctan z-\frac{1}{2}\ln\left(1+z^2\right)=\ln x+c$

Replacing z with $\frac{y}{x}$, we obtain $\arctan\frac{y}{x}-\frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right)=\ln x+c$

or $\arctan\frac{y}{x}=\ln\sqrt{x^2+y^2}+c$ as the general solution.

Example 5.10

Solve: $y'=\dfrac{xy}{x^2-y^2}$

Solution

We have $f(x,y)=\dfrac{xy}{x^2-y^2}$

$$f(tx,ty)=\frac{txty}{t^2x^2-t^2y^2}=\frac{t^2xy}{t^2\left(x^2-y^2\right)}=t^0\frac{xy}{x^2-y^2}=\frac{xy}{x^2-y^2}=f(x,y)$$

Then, $f(x,y)$ is a homogeneous function of degree 0.

To solve, let $\frac{y}{x}=z\Rightarrow y=zx$

$$\frac{dy}{dx}=z+x\frac{dz}{dx}\Rightarrow z+x\frac{dz}{dx}=\frac{xzx}{x^2-(zx)^2}\Rightarrow z+x\frac{dz}{dx}=\frac{x^2z}{x^2(1-z^2)}$$

$$\Rightarrow z+x\frac{dz}{dx}=\frac{z}{1-z^2}\Rightarrow z+x\frac{dz}{dx}-\frac{z}{1-z^2}=0\Rightarrow x\frac{dz}{dx}+z-\frac{z}{1-z^2}=0$$

$$\Rightarrow xdz+\left(z-\frac{z}{1-z}\right)dx=0\Rightarrow xdz+\left(\frac{z-z^3-z}{1-z^2}\right)dx=0\Rightarrow xdz-\frac{z^3}{1-z^2}dx=0$$

$$\Rightarrow\frac{xdz}{x\left(\frac{z^3}{1-z^2}\right)}-\frac{z^3}{1-z^2}\times\frac{dx}{x\left(\frac{z^3}{1-z^2}\right)}=0\Rightarrow\frac{dz}{\frac{z^3}{1-z^2}}-\frac{dx}{x}=0$$

$$\Rightarrow\frac{z^3}{1-z^2}dz-\frac{dx}{x}=0\Rightarrow\int\frac{1-z^2}{z^3}dz-\int\frac{dx}{x}=c$$

$$\Rightarrow\int\frac{dz}{z^3}-\int\frac{z^2}{z^3}dz-\int\frac{dx}{x}=c\Rightarrow-\frac{1}{2z^2}-\ln|z|-\ln|x|=c.$$

But $z=\dfrac{y}{x}$; then,

$$-\frac{1}{2\left(\frac{y}{x}\right)^2}-\ln\left|\frac{y}{x}\right|-\ln|x|=c \Rightarrow -\frac{x^2}{2y^2}-\ln\left|\frac{y}{x}\right|-\ln|x|=c$$

$$\Rightarrow -\frac{x^2}{2y^2}-\ln|y|+\ln|x|-\ln|x|=c$$

$$\Rightarrow -\frac{x^2}{2y^2}-\ln|y|=\ln k \text{ [where } c=\ln k\text{]}$$

$$\Rightarrow -\frac{x^2}{2y^2}=\ln k+\ln y \Rightarrow -\frac{x^2}{2y^2}=\ln ky \Rightarrow -x^2=2y^2\ln ky$$

$\therefore x^2=-2y^2\ln ky,\ k$ = constant as the general solution.

Application activity 5.3

Solve the following differential equations.

1) $\dfrac{dy}{dx}=\dfrac{x^2+xy}{xy+y^2}$

2) $2xy\dfrac{dy}{dx}=y^2-x^2$

3) $xy\dfrac{dy}{dx}=2y^2+4x^2$

4) $\dfrac{dy}{dx}=\dfrac{y-x+2}{y-x+1}$

5.2.3. Linear equations

Activity 5.4

Consider the equation $\dfrac{dy}{dx}+py=q \quad (1)$ where p and q are functions in x or constants. Suppose that the general solution of this equation has the form $y=uv$ where u and v are functions in x.

1. Differentiate y.
2. Substitute the expression obtained in 1) and value of y into equation (1).

3. In relation obtained in 2) by factoring out u and letting $\frac{dv}{dx}+pv=0$, solve for v. Give the value of v by taking the constant of integration to be 1.
4. Substitute this value of v, obtained in 3) into equation obtained in 2) (knowing that $\frac{dv}{dx}+pv=0$) and hence solve for u.

The most important type of differential equation is linear equation in which the derivative of highest order is a linear function of the lower order derivatives.

Thus, the general first order linear equation is $\frac{dy}{dx}+py=q$ where p and q are functions in x or constants.

From activity 5.4, the general solution to the differential equation $\frac{dy}{dx}+py=q$ where p and q are functions in x or constants, is $y=uv$ where $u=\int q\,e^{\int pdx}\,dx$ and $v=e^{-\int pdx}$.

Example 5.11

Solve: $y'-\frac{2}{x+1}y=(x+1)^3$

Solution

Let the solution be $y=uv$

$$\frac{dy}{dx}=\frac{udv}{dx}+\frac{vdu}{dx}$$

Now, the given differential equations becomes;

$$\frac{udv}{dx}+\frac{vdu}{dx}-\frac{2}{x+1}uv=(x+1)^3$$

$$u\frac{dv}{dx}-\frac{2}{x+1}uv+\frac{vdu}{dx}=(x+1)^3$$

$$u\left(\frac{dv}{dx}-\frac{2v}{x+1}\right)+\frac{vdu}{dx}=(x+1)^3$$

Taking $\frac{dv}{dx}-\frac{2v}{x+1}=0 \Rightarrow \frac{dv}{dx}=\frac{2v}{x+1}$ and $\frac{dv}{v}=\frac{2dx}{x+1}$

$$\Rightarrow \ln|v|=2\ln|x+1| \Rightarrow \ln|v|=\ln(x+1)^2 \Rightarrow v=(x+1)^2$$

Now, $u\left(\frac{dv}{dx}-\frac{2v}{x+1}\right)+\frac{vdu}{dx}=(x+1)^3$ becomes $(x+1)^2\frac{du}{dx}=(x+1)^3$

as $\frac{dv}{dx}-\frac{2v}{x+1}=0 \Rightarrow \frac{du}{dx}=x+1 \Rightarrow du=(x+1)dx \Rightarrow u=\int(x+1)dx$

$\Rightarrow u=\frac{(x+1)^2}{2}+c$ or $u=\frac{x^2}{2}+x+c$

Then,

$$y=(x+1)^2\left[\frac{(x+1)^2}{2}+c\right]=\frac{(x+1)^4}{2}+c(x+1)^2$$

Application activity 5.4

Solve:

1. $\frac{dy}{dx}+\frac{y}{x}=1,\ x>0$
2. $\frac{dy}{dx}+xy=x^3$
3. $(x+1)\frac{dy}{dx}-y=e^x(x+1)^2$
4. $\frac{dy}{dx}+2y\tan x=\sin x$

5.2.4. Particular solution

Activity 5.5

Consider the differential equation $\frac{dy}{dx}=x+4$

1. Find the general solution of this equation.
2. Find the value of the constant of integration if $y=4$ when $x=2$. Write down the new solution by replacing the constant of integration with its value.

We have already mentioned that the solution to a given differential equation is obtained by integration. If the solution contains one or more constant(s) of integration, then it is called a **general (primitive) solution**.

When more information is provided, the value of the constant can be determined and hence a **particular solution** can be written. Differential equations with more information are reffered to as **initial value problem**. In fact, in application, particular solutions are much more useful than general solutions.

Example 5.12

Solve: $y' - \frac{2}{x+1} y = (x+1)^3 \quad y|_{x=0} = 3$

Solution

We have seen that, in Example 5.11, the general solution for the equation is $y = \frac{(x+1)^4}{2} + c(x+1)^2$

Now, $y|_{x=0} = 3$ means that $y = 3$ for $x = 0$

Then,

$$3 = \frac{(0+1)^4}{2} c(0+1)^2 \Rightarrow 3 = \frac{1}{2} + c \Rightarrow c = 3 - \frac{1}{2} = \frac{5}{2}$$

Then, the particular solution for the given equation is

$$y = \frac{(x+1)^4}{2} + \frac{5}{2}(x+1)^2$$

Notice

There exists one and only one solution of the initial value problem $\frac{dy}{dx} = p(x)y$, $y(x_0) = y_0$ within a given interval.

Application activity 5.5

Find the solution satisfying the given conditions:

1. $\frac{dy}{dx} + 2y \tan x = \sin x, \quad y\left(\frac{\pi}{3}\right) = 0$
2. $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}, y(0) = 1$
3. $xy = (1-x^2)\frac{dy}{dx}, \; y(0) = 1$

4. $xy\frac{dy}{dx}=x^2-1,\ y(1)=2$

5. $\frac{dy}{dx}=e^{2x-y},\ y(0)=0$

5.3. Second order differential equations

Activity 5.6

Give two examples of second order differential equations with;

1. degree greater than 1
2. degree 1

The general second order linear differential equation is of the form

$$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=r(x)$$

Or more simply, $y''+p(x)y'+q(x)y=r(x)$ (1)

where $p(x)$, $q(x)$ and $r(x)$ are functions of x alone (or perhaps constants).

If $r(x)$ is identically zero, the differential equation reduces to the **homogeneous** equation;

$$y''+p(x)y'+q(x)y=0$$

If $r(x)$ is not identically zero, then the differential equation is said to be **non-homogeneous**.

If a second order differential equation cannot be written in the form (1), it is said to be **non-linear.**

For second order homogeneous linear equation $y''+py'+qy=0$, a general solution will be of the form $y=c_1y_1+c_2y_2$; a linear combination of two solutions involving two arbitrary constant c_1 and c_2. In this solution, y_1 and y_2 are called a **basis** of $y''+py'+qy=0$, where y_1 and y_2 are linearly independent.

An initial-value problem non-homogeneous consists of $y''+py'+qy=0$ and two initial conditions $y(x_0)=k_0$ and $y'(x_0)=k_1$, prescribing values k_0 and k_1 of the solution and its derivatives at the same given x.

5.3.1. Homogeneous linear equations with constant coefficients

Activity 5.7

1. Find the solution of the equation $y'+ky=0$, k is a constant
2. Substitute the solution obtained in 1) into the equation $y''+py'+qy=0$ and give the condition so that the solution obtained in 1) is a solution of $y''+py'+qy=0$. What can you say about the solution of $y''+py'+qy=0$?

Let $y''+py'+qy=0$ (1) be a homogeneous linear equation of second order (right hand side is equal to zero) where p and q are constants.

From Activity 5.7, the solution of the equation $y'+ky=0$ is a solution of $y''+py'+qy=0$ if the equation of the form $m^2+pm+q=0$ called the **characteristic auxiliary equation** is satisfied. The two roots m_1 and m_2 of this equation, i.e. the values of m are given by the quadratic formula

$$m_1, m_2 = \frac{-p \pm \sqrt{p^2-4q}}{2}.$$

Depending on the sign of the discriminant, $\Delta = p^2-4q$, we obtain:

Case 1: Two real roots if $p^2-4q>0$.

Case 2: A real double roots if $p^2-4q=0$.

Case 3: Complex conjugate roots if $p^2-4q<0$.

Case 1: Characteristic equation has two distinct real roots

It is clear that the roots m_1 and m_2 are distinct real numbers if and only if $\Delta>0$. In this case, we get the two solutions e^{m_1x} and e^{m_2x}.

Since the ratio $\frac{e^{m_1x}}{e^{m_2x}} = e^{(m_1-m_2)x}$ is not constant, these solutions are linearly independent and $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ is the general solution of (1) (c_1 and c_2 are arbitrary constants).

In this solution, $y_1 = c_1 e^{m_1 x}$ and $y_2 = c_2 e^{m_2 x}$ are called **basis** of $\frac{d^2 y}{dx^2} + p\frac{dy}{dx} + qy = 0$.

Example 5.13

Solve: $y'' + y' - 2y = 0$

Solution

The characteristic equation is $m^2 + m - 2 = 0$

$\Delta = 1 + 8 = 9 > 0$

$$m_1 = \frac{-1-3}{2} = -2,\ m_2 = \frac{-1+3}{2} = 1$$

Then the general solution is $y = c_1 e^{x} + c_2 e^{-2x}$.

Application activity 5.6

Solve the differential equations;

1. $\frac{d^2 y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$
2. $y'' + y' - 2y = 0$
3. $\frac{d^2 y}{dx^2} + \frac{dy}{dx} - 30y = 0$
4. $\frac{d^2 y}{dx^2} + 10\frac{dy}{dx} + 21y = 0$

Case 2: Characteristic equation has a real double root

Activity 5.8

Consider the differential equation:

$$\frac{d^2 y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

1. Determine one of its solutions.

2. Let $y_2 = xy_1$ where y_1 is a solution of $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$.

a) Substitute y by $y_2 = xy_1$ in the given differential equation; what can you conclude?

b) Verify if y_1 is linearly independent and deduce the general solution of $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=0$.

Hint:

General solution is a combination of two linearly independent solutions.

In characteristic equation $m^2+pm+q=0$, we know that the roots m_1 and m_2 are equal real numbers if and only if $\Delta=0$.

Here, we obtain only one solution $y_1 = e^{mx}$. However, from activity 5.8, the second linearly independent solution is $y_2 = xe^{mx}$ and the general solution of equation $y''+py'+qy=0$

is $y = c_1e^{mx}+c_2xe^{mx}=(c_1+c_2x)e^{mx}$.

Example 5.14

Solve: $y''-4y'+4y=0$

Solution

Characteristic equation is $m^2-4m+4=0$

$\Delta = 16-16=0$

$$m=\frac{4}{2}=2$$

The general solution is $y = c_1e^{2x}+c_2xe^{2x}$

Application activity 5.7

Solve:

1. $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=0$
2. $y''+6y'+9y=0$
3. $\frac{d^2y}{dx^2}+8\frac{dy}{dx}+16y=0$
4. $\frac{d^2y}{dx^2}-\frac{1}{3}\frac{dy}{dx}+\frac{1}{36}y=0$

Case 3: Characteristic equation has complex roots

Activity 5.9

Given the differential equation $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+25y=0$,

1. a) Write down the basis solution.

   b) Write its general solution.
2. Use Euler's formula to write the solution in two parts; one in function of cosine only and another in function of sine only.
3. Since the obtained solution is not real valued function, find the two real valued functions that are solutions of the given differential equation (real basis).
4. Hence, give the general solution of the differential equation $y''+py'+qy=0$.

When the characteristic equation has complex roots, the bases are $y_1=e^{(\alpha+i\beta)x}$ and $y_2=e^{(\alpha-i\beta)x}$ giving a general solution

$$y=c_1e^{(\alpha+i\beta)x}+c_2e^{(\alpha-i\beta)x}=e^{\alpha x}\left(c_1e^{i\beta x}+c_2e^{-i\beta x}\right)$$

The imaginary i is not always welcome here, so we use Euler's formula to put the solution into real form i.e.

$e^{i\beta x}=\cos\beta x+i\sin\beta x$ and $e^{-i\beta x}=\cos\beta x-i\sin\beta x$

Hence, $y=e^{\alpha x}\left[(c_1+c_2)\cos\beta x+(c_1-ic_2)\sin\beta x\right]$

From activity 5.9, the basis of real solution can be written as $y_1=e^{\alpha x}\cos\beta x$ and $y_2=e^{\alpha x}\sin\beta x$ and hence general solution is $y=e^{\alpha x}\left(A\cos\beta x+B\sin\beta x\right)$

Example 5.15

Find the general solution of $y''+2y'+5y=0$.

Solution

Characteristic equation is $m^2+2m+5=0$

$\Delta = 4 - 20 = -16 < 0 \ \sqrt{\Delta} = 4i$

$$m_1 = \frac{-2 - 4i}{2} = -1 - 2i, \ m_2 = \frac{-2 + 4i}{2} = -1 + 2i$$

$a = -1, \ b = 2$

The general solution is $y = e^{-x}\left(c_1 \cos 2x + c_2 \sin 2x\right)$.

Example 5.16

Solve the initial value problem

$$\begin{cases} y'' + 2y' + 2y = 0 \\ y(0) = 2 \\ y'(0) = -3 \end{cases}$$

Solution

Characteristic equation is $m^2 + 2m + 2 = 0$

$\Delta = 4 - 8 = -4 < 0 \ \sqrt{\Delta} = 2i$

$$m_1 = \frac{-2 - 2i}{2} = -1 - i, \ m_2 = \frac{-2 + 2i}{2} = -1 + i$$

$a = -1, \ b = 1$

The general solution is $y = e^{-x}\left(c_1 \cos x + c_2 \sin x\right)$.

Also,

$$y' = -e^{-x}\left(c_1 \cos x + c_2 \sin x\right) + e^{-x}\left(-c_1 \sin x + c_2 \cos x\right)$$

$$y(0) = 2 \Leftrightarrow 2 = e^{0}\left(c_1 \cos 0 + c_2 \sin 0\right)$$
$$\Rightarrow 2 = c_1$$

$$y'(0) = -3 \Leftrightarrow -3 = -e^{0}\left(c_1 \cos 0 + c_2 \sin 0\right) + e^{0}\left(-c_1 \sin 0 + c_2 \cos 0\right)$$
$$\Rightarrow -3 = -c_1 + c_2$$

$$\Rightarrow c_2 = -3 + c_1 = -3 + 2 = -1$$

The particular solution is $y = e^{-x}\left(2\cos x - \sin x\right)$.

Application activity 5.8

Solve the following differential equations.

1. $y'' + 4y' + 13y = 0$
2. $y'' + 4y' + 5y = 0$
3. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$
4. $y'' + 4y' + 13y = 0,\ y(0) = 0 \text{ and } y\left(\dfrac{\pi}{2}\right) = 1$
5. $2y'' + y' - 10y = 0,\ y(0) = 0 \text{ and } y(1) = 1$

5.3.2. Non-homogeneous linear equations with constant coefficients

Activity 5.10

State the type of the following differential equations and solve if possible.

1. $\dfrac{dy}{dx} - \dfrac{y}{x+1} = e^x(x+1)$
2. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} = 5y$
3. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 5y = x$

The general solution of the second order non-homogeneous linear equation $y'' + py' + qy = r(x)$ can be expressed in the form $y = \overline{y} + y^*$ where $\overline{y}$ is any specific function that satisfies the non-homogeneous equation, and $\overline{y} = c_1y_1 + c_1y_1$ is a general solution of the corresponding homogeneous equation $y'' + py' + qy = 0$.

The term $\overline{y} = c_1y_1 + c_1y_1$ is called the **complementary solution** (or the **homogeneous solution**) of the non-homogeneous equation.

The term y^* is called the **particular solution** (or the **non-homogeneous solution**) of the same equation and its form depends on the type of $r(x)$.

The right hand side is a product of the form $r(x) = Pe^{\alpha x}$

Activity 5.11

Consider the equation $y'' - 2y' + y = e^x$.

1. Find the general solution, say $\overline{y}$, of $y'' - 2y' + y = 0$.
2. Express the right hand side of the given equation in the form $r(x) = Pe^{\alpha x}$. Suppose that the given equation has particular solution $y^* = x^k Q(x) e^{\alpha x}$

   where;

   α is the coefficient of x in $e^{\alpha x}$ in the right hand side of the given equation, k is the number of roots of the characteristic equation obtained in 1) equals to α, and $Q(x)$ is the polynomial with the same degree as the degree of the polynomial found in right hand side of the given equation.

   Write down y^*.
3. Substitute the value of y^* in the given equation to find the new expression for y^*.

If the right hand side of the equation $y'' + py' + qy = r(x)$ is $r(x) = Pe^{\alpha x}$ where P is a polynomial, we take the particular solution to be

$$y^* = x^k Q_n(x) e^{\alpha x},\ Q_n = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ... + a_n$$

Here, k - is the number of roots of the associated homogeneous equation equals to α.

n - degree of $Q(x)$, the same as degree of $P(x)$ on the right hand side.

α - coefficient of x in $e^{\alpha x}$ in the right hand side.

Three cases arise

- If α is not a root of characteristic equation $k = 0$.
- If α is a simple root of characteristic equation $k = 1$.
- If α is a double root of characteristic equation $k = 2$.

Note that the simple root or double root in the last 2 cases must be real numbers.

Example 5.17

Find the general solution of $y''+y=e^x$.

Solution

Characteristic equation:

$$m^2+0m+1=0$$
$$\Delta=0-4=-4$$
$$m_1=\frac{0-2i}{2}=-i,\ m_2=\frac{0+2i}{2}=i$$

$\bar{y}=c_1\cos x+c_2\sin x$, which is the general solution of the homogeneous equation.

$\alpha=1$ is not a solution of the characteristic equation, so $k=0$;

Taking $y^*=Ae^x$ and $Q(x)=A$ as $P(x)=1$.

$$y^*''=Ae^x$$

Substituting the expression into the given equation gives

$\Leftrightarrow 2Ae^x=e^x$. Thus $\Leftrightarrow 2A=1$

Or $A=\frac{1}{2}\Rightarrow y^*=\frac{1}{2}e^x$.

The general solution of the given equation is

$$y=\bar{y}+y^*\Rightarrow y=c_1\cos x+c_2\sin x+\left(\frac{1}{2}e^x\right).$$

Example 5.18

Find the general solution of $y''-7y'+6y=(x-2)e^x$

Solution

Characteristic equation: $m^2-7m+6=0$

$$\Delta=49-24=25$$

$$m_1 = \frac{7-5}{2} = 1,\ m_2 = \frac{7+5}{2} = 6$$

We see that $\alpha = 1$ which is one of the roots of characteristic equation, so $k = 1$

Then $y^* = x(Ax+B)e^x$,

$Q(x) = Ax + B$ as $P(x) = x - 2$

But $\overline{y} = c_1e^x + c_2e^{6x}$

$$y^* = x\left(Ax+B\right)e^x = \left(Ax^2+Bx\right)e^x$$

$$y^{*/} = \left(2Ax+B\right)e^x + \left(Ax^2+Bx\right)e^x$$

$$y^{*//} = 2Ae^x + \left(2Ax+B\right)e^x + \left(2Ax+B\right)e^x + \left(Ax^2+Bx\right)e^x$$

Substituting these expressions into the given equation gives;

$$2Ae^x + \left(2Ax+B\right)e^x + \left(2Ax+B\right)e^x + \left(Ax^2+Bx\right)e^x$$
$$-7\left(2Ax+B\right)e^x - 7\left(2Ax^2+Bx\right)e^x + 6\left(Ax^2+Bx\right)e^x = \left(x-2\right)e^x$$

$$\Rightarrow 2A + 2Ax + B + 2Ax + B + Ax^2 + Bx - 14Ax$$
$$-7B - 7Ax^2 - 7Bx + 6Ax^2 + 6Bx = x - 2$$

$$\Rightarrow x^2\left(0A\right) + x\left(-10A + 0B\right) + 2A - 5B = x - 2$$

$$\Rightarrow -10Ax + 2A - 5B = x - 2$$

$$\begin{cases} -10A = 1 \\ 2A - 5B = -2 \end{cases}$$

$$A = -\frac{1}{10},\ B = \frac{9}{25}$$

$y^* = x\left(-\frac{x}{10} + \frac{9}{25}\right)e^x$ and $y = c_1e^x + c_2e^{6x} + x\left(-\frac{x}{10} + \frac{9}{25}\right)e^x$

Application activity 5.9

Solve the following differential equations.

1. $y''+6y'+9y=5e^{3x}$
2. $y''-3y'+2y=e^{3x}$
3. $y''+3y+2y=3e^{2x}$
4. $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=e^{3x}$
5. $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=e^{3x}$

The right hand side is of the form $r(x)=Pe^{\alpha x}\cos\beta x+Qe^{\alpha x}\sin\beta x$

Activity 5.12

Consider the equation $y''+4y=\cos 2x$

1. Find the general solution, say $\overline{y}$, of $y''+4y=0$.
2. Write the right hand side of the given equation in the form $Pe^{\alpha x}\cos\beta x+Qe^{\alpha x}\sin\beta x$ where P and Q are polynomials. Suppose that the given equation has particular solution $y^*=x^r\left(ue^{\alpha x}\cos\beta x+ve^{\alpha x}\sin\beta x\right)$

   where;

   a) $r=0$ if $\alpha+i\beta$ is not a root of characteristic equation,

   b) $r=1$ if $\alpha+i\beta$ is a root of characteristic equation,

   c) u and v are polynomials in x of degree equal to the highest degree of P and Q.

   Write down y^*
3. Substitute the value of y^* into the given equation to find the new expression for y^*.

If the right hand side of the equation $y''+py'+qy=r(x)$ is $r(x)=Pe^{\alpha x}\cos\beta x+Qe^{\alpha x}\sin\beta x$ where P and Q are polynomials, two cases arise:

- If $\alpha+i\beta$ is not a root of characteristic equation, the particular solution is

  $$y^*=Ue^{\alpha x}\cos\beta x+Ve^{\alpha x}\sin\beta x$$

- If $\alpha + i\beta$ is a root of characteristic equation, the particular solution becomes

$$y^* = x\left[Ue^{\alpha x}\cos\beta x + Ve^{\alpha x}\sin\beta x\right]$$

In all cases, U and V are polynomial of degree which is equal to the highest degree of P and Q.

Example 5.19

Find the general solution of $y'' + 2y' + 5y = 2\cos x$.

Solution

Characteristic equation: $m^2 + 2m + 5$

$\Delta = 4 - 20 = -16$

$$m_1 = \frac{-2-4i}{2} = -1-2i,\ m_2 = \frac{-2+4i}{2} = -1+2i$$

$$\bar{y} = e^{-x}\left(c_1\cos 2x + c_2\sin 2x\right)$$

Here, $\alpha = 0,\ \beta = 1$ because the right hand side can be written as $2e^{0x}\cos x$.

We see that $\alpha + i\beta = 0 + i$ is not a solution of characteristic equation, then,

$$y^* = A\cos\beta x + B\sin\beta x$$

$$= A\cos x + B\sin x$$

$$y^{*/} = -A\sin x + B\cos x$$

$$y^{*//} = -A\cos x - B\sin x$$

$$\Rightarrow -A\cos x - B\sin x + 2\left(-A\sin x + B\cos x\right) + 5\left(A\cos x + B\sin x\right) = 2\cos x$$

$$\Rightarrow -A\cos x - B\sin x - 2 - A\sin x + 2B\cos x + 5A\cos x + 5B\sin x = 2\cos x$$

$$\Rightarrow \cos x\left(-A + 2B + 5A\right) + \sin x\left(-B - 2A + 5B\right) = 2\cos x$$

$$\begin{cases} -A + 2B + 5A = 2 \\ -B - 2A + 5B = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 4A+2B=2 \\ -2A+4B=0 \end{cases} \quad \Rightarrow \begin{cases} 2A+B \quad =1 \\ -2A+4B=0 \end{cases}$$

$$5B=1$$

$B=\dfrac{1}{5}$, and $A=\dfrac{2}{5}$

$$y^*=\frac{2}{5}\cos x+\frac{1}{5}\sin x$$

Thus; $y=e^{-x}\left(c_1\cos 2x+c_2\sin 2x\right)+\dfrac{2}{5}\cos x+\dfrac{1}{5}\sin x$

Alternative method: Variation of parameters

We know that the general solution of the characteristic equation associated to the equation $y''+py'+qy=r(x)$ is found to be $\overline{y}=c_1y_1(x)+c_2y_2(x)$.

From $\overline{y}=c_1y_1\left(x\right)+c_2y_2\left(x\right)$, we can get particular solution y^* as follows:

- We determine $W\left(y_1,y_2\right)$ known as **Wronskian** of two functions y_1 and y_2 defined by

  $W\left(y_1,y_2\right)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}\neq 0$,since y_1 and y_2 are linearly independent.

- We determine $v_1=\displaystyle\int\frac{-y_2r(x)}{W(y_1,y_2)}$, and $v_2=\displaystyle\int\frac{y_1r(x)}{W(y_1,y_2)}$

where $r(x)$ is the right hand side of the given equation.

Then the particular solution y^* is given by $y^*=v_1(x)y_1(x)+v_2(x)y_2(x)$.

Therefore, the general solution is $y=\overline{y}+y^*$

$$\Rightarrow y=c_1y_1(x)+c_2y_2(x)+v_1(x)y_1(x)+v_2(x)y_2(x)$$

Example 5.20

Find the general solution of $y''-2y'-3y=e^{2x}$

Solution

Characteristic equation:

$$m^2 - 5m + 6 = 0$$

$$\Delta = 4 + 12 = 16$$

$$m_1 = \frac{2-4}{2} = -1,\ m_2 = \frac{2+4}{2} = 3$$

The complementary solution for this differential equation is $\overline{y} = c_1 e^{-x} + c_2 e^{3x}$

So, we have

$y_1 = e^{-x}, y_2 = e^{3x} \Rightarrow y'_1 = -e^{-x}$ and $y_2' = 3e^{3x}$.

The Wronskian of these two functions is

$$W(y_1, y_2) = \begin{vmatrix} e^{-x} & e^{3x} \\ -e^{-x} & 3e^{3x} \end{vmatrix} = 3e^{-x}e^{3x} + e^{-x}e^{3x} = 4e^{2x}$$

Now; $v_1 = \int -e^{3x} \frac{e^{2x}}{4e^{2x}} dx = -\frac{1}{4}\int e^{3x} dx = -\frac{1}{12} e^{3x}$

Also; $v_2 = \int e^{-x} \frac{e^{2x}}{4e^{2x}} dx = \frac{1}{4}\int e^{-x} dx = -\frac{1}{4} e^{-x}$

Particular solution is,

$$y^* = v_1 y_1 + v_2 y_2 = \left(-\frac{1}{12}e^{3x}\right)e^{-x} + \left(-\frac{1}{4}e^{-x}\right)e^{3x} = -\frac{1}{3}e^{2x}$$

Thus the general solution is

$$y = c_1 e^{-x} + c_2 e^{3x} - \frac{e^{2x}}{3}$$

Example 5.21

Find the general solution of $y'' - 5y' + 6y = e^x \sin x$.

Solution

Characteristic equation:

$$m^2 - 5m + 6 = 0 \qquad \Delta = 25 - 24 = 1$$

$$m_1 = \frac{5-1}{2} = 2, m_2 = \frac{5+1}{2} = 3$$

The complementary solution for this differential equation is $\bar{y} = c_1 e^{2x} + c_2 e^{3x}$

So, we have

$y_1 = e^{2x}$, $y_2 = e^{3x} \Rightarrow y'_1 = 2e^{2x}$ and $y'_2 = 3e^{3x}$.

The Wronskian of these two functions is

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{2x}e^{3x} - 2e^{2x}e^{3x} = e^{5x}$$

Now, $v_1 = \int -e^{3x} \frac{e^x \sin x}{e^{5x}} dx = -\int e^{-x} \sin x dx$

Integrating by parts, gives

$$v_1 = \frac{1}{2} e^{-x} \sin x + \frac{1}{2} e^{-x} \cos x = e^{-x} \left(\sin x + \frac{1}{2} \cos x \right)$$

Also, $v_2 = \int e^{2x} \frac{e^x \sin x}{e^{5x}} dx = \int e^{-2x} \sin x dx$

Integrating by parts, gives

$$v_2 = \int e^{-2x} \sin x dx = -\frac{2}{5} e^{-2x} \sin x - \frac{1}{5} e^{-2x} \cos x = e^{-2x} \left(-\frac{2}{5} \sin x - \frac{1}{5} \cos x \right)$$

Particular solution becomes,

$$y^* = v_1 y_1 + v_2 y_2$$

$$= e^x \left(\sin x + \frac{1}{2} \cos x \right) + e^x \left(-\frac{2}{5} \sin x - \frac{1}{5} \cos x \right) = \frac{e^x}{10} (3 \cos x + \sin x)$$

Thus the general solution is

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{e^x}{10} (3 \cos x + \sin x)$$

Example 5.22

Find a general solution to the following differential equation

$$\frac{d^2 y}{dx^2} - 2 \frac{dy}{dx} + y = \frac{e^x}{x^2 + 1}$$

Solution

We first need the complementary solution for this differential equation.

Characteristic equation:

$m^2 - 2m + 1 = 0 \Leftrightarrow (m-1)^2 = 0$, then $m = 1$.

The complementary solution for this differential equation is $\bar{y} = c_1 e^x + c_2 x e^x$

So, we have

$y_1 = e^x, y_2 = xe^x$ and $y_1' = e^x, y_2' = e^x + xe^x$

Thus,

$$W(y_1, y_2) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^x\left(e^x + xe^x\right) - e^x\left(xe^x\right) = e^{2x};$$

$$v_1 = \int -xe^x \frac{e^x}{e^{2x}\left(x^2+1\right)} dx = -\int \frac{x}{x^2+1} dx = -\frac{1}{2}\ln\left(x^2+1\right);$$

$$v_2 = \int e^x \frac{e^x}{e^{2x}\left(x^2+1\right)} dx = \int \frac{dx}{x^2+1} = \tan^{-1} x$$

The particular solution is $y^* = -\frac{1}{2}e^x \ln\left(x^2+1\right) + xe^x \tan^{-1} x$

The general solution is,

$$y = c_1 e^x + c_2 xe^x - \frac{1}{2}e^x \ln\left(x^2+1\right) + xe^x \tan^{-1} x$$

Example 5.23

Find the general solution of $y'' + y = e^x$

Solution

Characteristic equation: $m^2 + 1 = 0 \Rightarrow m = \pm i$

$\bar{y} = c_1 \cos x + c_2 \sin x$

Let $y^* = v_1 \cos x + v_2 \sin x$,

$y_1 = \cos x,\ y_2 = \sin x,\ r(x) = e^x \Rightarrow y'_1 = -\sin x,\ y'_2 = \cos x$

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$v_1 = \int -\sin x e^x dx = -\frac{e^x}{2}(\sin x - \cos x)$$

$$v_2 = \int \cos x e^x dx = \frac{e^x}{2}(\cos x + \sin x)$$

$$y^* = -\frac{e^x}{2}(\sin x - \cos x)\cos x + \frac{e^x}{2}(\cos x + \sin x)\sin x = \frac{1}{2}e^x$$

Thus; $y = c_1 \cos x + c_2 \sin x + \frac{1}{2}e^x$

Application activity 5.10

Find the general solution of:

1. $y'' - 2y' + y = x\sin x$
2. $y'' - 2y' + y = xe^x \sin x$
3. $y'' - y = x\sin 3x + \cos x$
4. $y'' + y = \csc x$
5. $y'' + y = \tan x$

5.4. Applications

There are a number of well-known applications of first order equations which provide classic prototypes for mathematical modeling. These mainly rely on the interpretation of $\frac{dy}{dt}$ as a rate of change of a function y with respect to time t.

In everyday life, there are many examples of situations that involve rates of change. These include; speed of moving particles, growth and decay of populations and materials, heat flow, fluid flow, and so on. In each case, we can construct models of varying degrees of sophistication to describe given situations.

5.4.1. Newton's law of cooling

Activity 5.13

Using the library or internet if available, show how differential equations are used in Newton's law of cooling and hence solve the following problem:

Suppose that you are in hurry to go out but you want to drink a cup of hot coffee before you go. The initial temperature of the coffee is 90^0C and you can start to drink the coffee when its temperature is 45^0C. The temperature of the room (ambient air) is 20^0C.

Formulate a model and find out how long you will have to wait.

What assumptions and simplifications have you made?

Newton's law of cooling states that the rate at which an object cools is proportional to the difference between the temperature at the surface of the body, and the ambient air temperature.

Thus, if $T°C$ is the surface temperature at time t and $T_a°C$ is the ambient temperature, then, we can write; $\frac{dT}{dt} = -\lambda\left(T - T_a\right)$

where $\lambda > 0$ is some experimentally determined constant of proportionality, and T_0 is the initial temperature of the body of interest.

Example 5.24

Solve the following equation

$$\frac{dT}{dt} = -\lambda\left(T - T_a\right)$$

where $\lambda > 0$ to give the temperature at $t > 0$.

Solution

$$\frac{dT}{dt} = -\lambda\left(T - T_a\right) \Leftrightarrow \frac{dT}{T - T_a} = -\lambda dt$$

Integrating both sides yields

$$\ln(T-T_a) = -\lambda t + c$$

$$\Rightarrow \ln(T-T_a) = \ln e^{-\lambda t} + c$$

$$\Rightarrow \ln\left(\frac{T-T_a}{e^{-\lambda t}}\right) = c \Leftrightarrow \frac{T-T_a}{e^{-\lambda t}} = e^c$$

$\Rightarrow T - T_a = ke^{-\lambda t}$ where $e^c = k \in \mathbb{R}$ $\Leftrightarrow T = T_a + ke^{-\lambda t}$

5.4.2. Electrical Circuits

Activity 5.14

Using the library or internet if available, show how differential equation are used in electric circuit and hence solve the following problem:

The current i in an electric circuit having a resistance R and inductance L in series with a constant voltage source E is given by the differential equation $E - L\left(\frac{di}{dt}\right) = Ri$.

a) Solve the equation and find i in terms of time t given that when $t = 0, i = 0$.

b) Find the value of i that corresponds to $t = 3\frac{L}{R}$ and

show that it is about 95% of the steady state value $I = \frac{E}{R}$.

c) Determine approximately the percentage of the steady state current that will be flowing in the circuit 2 times constants after the switch is closed (i.e., when $t = 2\frac{L}{R}$)

In the R−L series circuit shown in figure 5.1, the supply p.d., E, is given by

$$E = V_R + V_L$$

$V_R = iR$ and $V_L = L\frac{di}{dt}$

Hence, $E = iR + L\frac{di}{dt}$ from which $E - L\frac{di}{dt} = iR$

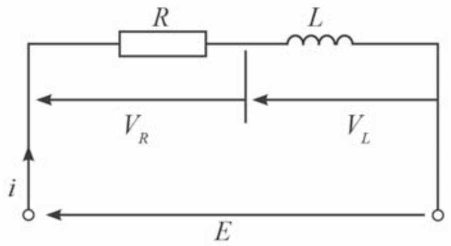


Figure 5.1. *R-L series circuit*

The corresponding solution is $i = \dfrac{E}{R}\left(1 - e^{-\frac{Rt}{L}}\right)$ which represents the law of growth of current in an inductive circuit as shown in figure 5.2

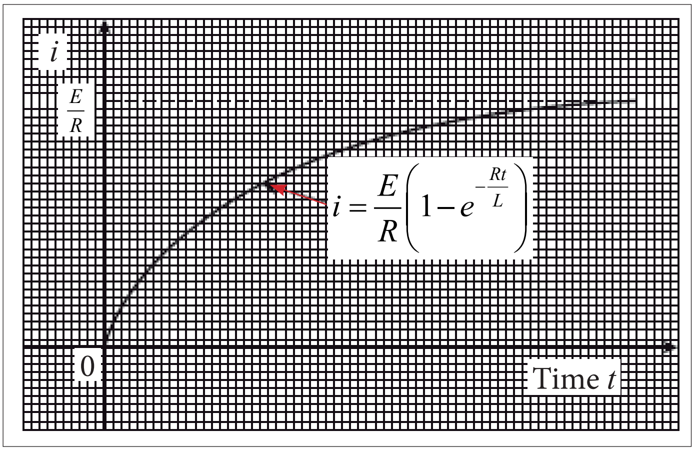


Figure 5.2. *Law of growth of current*

The growth of the current in the RL circuit, is the current's steady-state value. The number $t = \dfrac{L}{R}$ is the **time constant of the circuit**. The current gets to within 5% of its steady-state value in 3 times constant.

Example 5.25

The p.d., V, between the plates of a capacitor C charged by a steady voltage E through a resistor R is given by the equation

$$CR\frac{dV}{dt} + V = E.$$

a) Solve the equation for V given that at $t = 0,\ V = 0$.

b) Calculate V, correct to 3 significant figures, when $E = 25V$, $C = 20\times10^{-6}F$, $R = 200\times10^{3}\Omega$ and $t = 3.s$.

Solution

$$CR\frac{dV}{dt}+V=E$$

$$CR\frac{dV}{dt}=E-V \Leftrightarrow \frac{dV}{E-V}=\frac{1}{CR}dt \Leftrightarrow \frac{dV}{E-V}=\frac{1}{CR}dt$$

Integrating yields;

$$\Rightarrow -\ln(E-V)=\frac{t}{CR}+k$$

But $\ln\frac{1}{E}=k$, when $t=0; V=0$

$$\Rightarrow -\ln(E-V)=\frac{t}{CR}+\ln\frac{1}{E}$$

$$\ln\frac{1}{E-V}-\ln\frac{1}{E}=\frac{t}{CR}$$

$$\ln\frac{E}{E-V}=\frac{t}{CR}\Rightarrow\frac{E-V}{E}=e^{-\frac{t}{CR}}$$

$$E-V=Ee^{-\frac{t}{CR}}\Rightarrow V=E-Ee^{-\frac{t}{CR}}$$

$V=E-Ee^{-\frac{t}{CR}}$ or $V=E\left(1-e^{-\frac{t}{CR}}\right)$

Application activity 5.11

1. The population of a colony of rabbits in a park increases at a rate proportional to the population. Initially, there were ten rabbits in the park. When the population is 100 rabbits, the colony is increasing at a rate of seven rabbits per month. Form a differential equation for the population increase and solve it.

2. The current in an electric circuit is given by the equation

   $Ri+L\frac{di}{dt}=0$, where R and L are constants.

   Show that $i=Ie^{-\frac{Rt}{L}}$ given that $i=I$ when $t=0$.

3. The body of a murder victim was discovered in the early hours of the morning at 2:00 a.m. The police surgeon arrived at 2:30 a.m. and immediately took the temperature of the body, which was 34.8^0C. One hour later the temperature of the body was 34.1^0C. Surrounding temperature was constant at 32.2^0C. If the normal body temperature is 37^0C,

   a) Formulate a differential equation model for the temperature of the body as a function of time.

   b) Solve the differential equation.

   c) Use your solution in b) to estimate the time of death.

4. Charge, Q (coulombs) at time, t (seconds) is given by the differential equation; $R\frac{dQ}{dt}+\frac{Q}{C}=0$, where C is the capacitance in farads and R the resistance in ohms.

   a) Solve the equation for Q given that $Q=Q_o$ when $t=0$.

   b) A circuit possesses a resistance of $250\times10^3\Omega$ and a capacitance of $8.5\times10^{-6}F$, and after 0.32 seconds the charge falls to $8.0C$. Determine the initial charge and the charge after 1 second, in each case correct to 3 significant figures.

5. A differential equation relating the difference in tension T, pulley contact angle θ and coefficient of friction μ is $\frac{dT}{d\theta}=\mu T$.

   When θ =0, T =150 N, and $\mu=0.30$ as slipping starts.

   Determine the tension at the point of slipping when θ =2 radians. Also, determine the value of θ when T is 300 N.

Unit summary

1. Definition and classification

An equation involving one or more differential coefficient(s) i.e.
$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{dr}{dt}$ etc. is called
a **differential equation**.

Order of the highest derivative of function y that appears in a differential equation is said to be the **order of differential equation**.

The general ordinary differential equation of the n^{th} order is $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots, \frac{d^ny}{dx^n}\right) = 0$, for derivatives
$F\left(x, y, y', y', \ldots\ldots, y^{(n)}\right) = 0$

2. First order differential equations

The general differential equation of the 1st order is $F\left(x, y, \frac{dy}{dx}\right) = 0$ or $\frac{dy}{dx} = f(x, y)$

The simplest is that in which the variables are separable: $\frac{dy}{dx} = g(x)h(y)$.

A homogeneous equation of degree 0 can be expressed as a function of
$z = \frac{y}{x}$

The general solution to the equation $\frac{dy}{dx} + py = q$
where p and x are functions in x or constants, is $y = uv$ where
$u = \int q\, e^{\int pdx}\, dx$ and $v = e^{-\int pdx}$.

3. Second order differential equations

The general second order linear differential equation is of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

Let $y'' + py' + qy = 0$ be a homogeneous linear equation of second order (right hand side is equal to zero) where p and q are constants.

The equation $m^2 + pm + q = 0$ is called the **characteristic auxiliary equation**

- If characteristic equation has two distinct real roots, then, $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$ is the general solution of $y'' + py' + qy = 0$.
- If characteristic equation has a real double root then, $y = c_1 e^{mx} + c_2 x e^{mx}$ is the general solution of $y'' + py' + qy = 0$.
- If characteristic equation has complex roots then, $y = e^{ax}\left(c_1 \cos bx + c_2 \sin bx\right)$ is the general solution of $y'' + py' + qy = 0$.

Let $y'' + py' + qy = r(x)$ (1) be a non-homogeneous linear equation of second order (right hand side is different from zero) where p and q are real numbers.

If the right hand side of the equation $y'' + py' + qy = r(x)$ is $r(x) = Pe^{\alpha x}$ where p is a polynomial, we take the particular solution to be

$= x^k Q_n(x) e^{\alpha x}$, $Q_n = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots$ -

Here: k - is the number of roots of the associated homogeneous equation equals to α .

n - degree of $Q(x)$, the same as degree of $P(x)$ in right hand side.

α - coefficient of x in $e^{\alpha x}$ in the right hand side

- If α is not a root of characteristic equation $k = 0$
- If α is a simple root of characteristic equation $k = 1$.
- If α is a double root of characteristic equation $k = 2$.

Note that the simple root or double root in the last 2 cases must be real numbers.

If the right hand side of the equation $y'' + py' + qy = r(x)$ is $r(x) = Pe^{\alpha x}\cos\beta x + Qe^{\alpha x}\sin\beta x$ where P and Q are polynomials, two cases arise:

- If $\alpha + i\beta$ is not a root of characteristic equation, the particular solution is $y^* = Ue^{\alpha x}\cos\beta x + Ve^{\alpha x}\sin\beta x$

- If $\alpha + i\beta$ is a root of characteristic equation, the particular solution becomes $y^* = x\left[Ue^{\alpha x}\cos\beta x + Ve^{\alpha x}\sin\beta x\right]$

In all cases, U and V are polynomial of degree which is equal to the highest degree of P and Q.

Alternative method: Variation of parameters

Assume that the general solution of the characteristic equation associated with the equation $y'' + py' + qy = r(x)$ is found to be $\overline{y} = c_1 y_1(x) + c_2 y_2(x)$

To get particular solution:

From $\overline{y} = c_1 y_1(x) + c_2 y_2(x)$, we determine $W(y_1, y_2)$ known as Wronskian of two linearly independent functions y_1 and y_2 defined by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

Find out $v_1 = \int \frac{-y_2 r(x)}{W(y_1, y_2)}$, and $v_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)}$

where $r(x)$ is the right hand side of the given equation.

Here, $W(y_1, y_2)$ must be different from zero as y_1 and y_2 are linearly independent.

Hence, the particular solution is $y^* = v_1(x)y_1(x) + v_2(x)y_2(x)$

The general solution is $y = \overline{y} + y^*$

4. Applications

There are a number of well-known applications of first order equations which provide classic prototypes for mathematical modeling. These mainly rely on the interpretation of $\frac{dy}{dt}$ as a rate of change of a function y with respect to time t. In everyday life, there are many examples of the importance of rates of change – speed of moving particles, growth and decay of populations and materials, heat flow, fluid flow, and so on. In each case we can construct models of varying degrees of sophistication to describe given situations.

End of unit assessment

1. Solve the following differential equations.

   a) $y' = x,\ y(0) = 1$

   b) $xy' = x^2 + 1$, $y(1) = 0, y(2) = 1$

   c) $y'' = \cos x,\ y(0) = 0,\ y(\pi) = 1$

   d) $y' = e^{2x} - y$ e) $(x-1)y' = 3x^2 - y$

   f) $(x-1)y' = 3x^2 - y$ g) $xy' - 2y = x^3 e^{-2x}$

2. Solve the following equations.

   a) $\dfrac{dy}{dx} = e^{2x-3y}$ b) $\dfrac{dy}{dx} = y \ln x$

   c) $4xy\dfrac{dy}{dx} = y^2 - 1$ d) $x\dfrac{dy}{dx} = 3x + 2y$

   e) $2xy\dfrac{dy}{dx} = y^2 - x^2$ f) $\dfrac{dy}{dx} = \dfrac{x+2y}{3x+y}$

   g) $\dfrac{dy}{dx} + 3y = e^{2x}\left(1 - x^2\right)$ h) $\dfrac{dy}{dx} + 2xy = x\sqrt{1-x^2}$

3. Find the particular solutions of the following initial value problems.

   a) $2y(1-x) + x(1+y)\dfrac{dy}{dx} = 0,$ given $x = 1$ when $y = 1$

   b) $y - x = x\dfrac{dy}{dx}, y(1) = 2$

   c) $x^2 - 3y^2 + 2xy\dfrac{dy}{dx}, y\Big|_{x=1} = 3$

   d) $\dfrac{dy}{dx} - y = e^x, y\big|_{x=1} = 0$ $y = e^x(x-1)$

   e) $\dfrac{dy}{dx} = y\tan x + 1, y\big|_{x=0} = 2$ $y = \tan x + 2\sec x$

   f) $\dfrac{dy}{dx} + y = \left(x^2 + 1\right), y(0) = 0$ $y = x^2 + 1 - e^{-x}$

4. Solve the following differential equations.

a) $\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=0$

b) $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+5y=0$

c) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+4y=0$

d) $\frac{d^2y}{dx^2}+4y=0$

e) $\frac{d^2y}{dx^2}-9y=0$

5. Obtain the general solution of the following in homogeneous equations.

a) $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+4y=3-2x$

b) $\frac{d^2y}{dx^2}+4y=xe^{2x}$

c) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y=2\cos 3x$

d) $\frac{d^2y}{dx^2}+4y=e^{-x}\cos x$

e) $\frac{d^2y}{dx^2}+4y=e^{x}+\sin 2x$

f) $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+3y=2xe^{3x}+e^{3x}\cos 2x$

6. For each of the following equations, determine the particular solutions for initial value problem.

a) $2y''+y'-10y=0,\quad y(0)=0,\ y'(0)=1$

b) $y''+4y'+13y=0,\quad y(0)=0,\ y'\left(\frac{\pi}{2}\right)=1$

7. Workout the solution for each of the following second order equations, with the specified conditions.

a) $y''+2y'+y=x+2,\quad y(0)=0,\ y'(0)=0$

b) $y''+4y=x+1,\ y(0)=0,\ y\left(\frac{\pi}{4}\right)=\frac{1}{4}$

c) $y''+y=\sin 2x,\ y(0)=0,\ y'(0)=0$

d) $y''-4y'+3y=3x,\ y(0)=0,\ y'(0)=0$

e) $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 10y = e^{2x} + 20;$ when $x = 0,\ y = 0$

and $\dfrac{dy}{dx} = -\dfrac{1}{3}$

f) $2\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = 6e^x \cos x;$ when $x = 0,\ y = -\dfrac{21}{29}$

and $\dfrac{dy}{dx} = -6\dfrac{20}{29}$

g) $y'' - 2y' + 2y = 3e^x \cos 2x,\ y(0) = 2,\ y'(0) = 3$

8. As a radioactive substance decays it loses its mass at a rate proportional to its mass at the present time. Write down a differential equation to model this statement.

9. An individual in a population of 1,500 people working in a company becomes infected with a virus. It is assumed that the rate at which the virus spreads throughout the company is proportional to the number of people infected, P, and to the number of people not infected. Form a differential equation to model the number of people infected as a function of time.

10. The rate of growth of sunflower after germination is initially proportional to its height. The growth rate is 2.5 m per day when its height is 10 m. In modeling the growth, it is assumed that the initial height is 2 cm. Formulate a problem consisting of the differential equation and initial condition to find the height of the sunflower at any time.

11. The charge $q(t)$ in an RC circuit satisfies the linear differential equation $q' + \dfrac{1}{RC}q = \dfrac{1}{R}E(t)$

a) Solve for the charge in the case that $q(0) = q_o$, constant. Evaluate the constant of integration by using the condition $q(0) = q_o$.

b) Determine $\lim\limits_{x \to +\infty} q(t)$ and show that this limit is independent of q_o

c) Determine at what time $q(t)$ is within 1% of its steady-state value (the limiting value requested in b)).

12. The rate at which a body cools is proportional to the difference between the temperature of the body and that of surrounding air. If a body in air at 25^0C will cool from 100^0C to 75^0C in one minute, find its temperature at the end of a further two minutes.

13. Water at temperature 100^0C cools in 10 minutes to 88^0C in a room of temperature 25^0C. Find the temperature of water after 20 minutes.

Unit 6

Intersection and Sum of Subspaces

Introductory activity

The mathematical concepts of vector spaces are formed according to the following natural rules:

a) Each vector can be magnified or shrank by a factor by simply changing the size but not the direction.

b) Vectors can be added. Two forces, for example, applied at the same time and at the same spot will have the same effect of a certain single force.

c) Two identical vectors added together would be a vector in the same direction, but twice the size.

Refer to what you studied in previous levels and give 2 examples in a), b) and in c).

Likewise, different vector spaces can be added to make a new vector space. For example, from $H=\left\{\left(a-2b,3a+b,2a+b\right):a,b\in\mathbb{R}\right\}$ and $K=\left\{\left(3b,b,2b\right):b\in\mathbb{R}\right\}$, you can find the sum and intersection of H and K. In this unit, we shall see such kind of operations.

Objectives

By the end of this unit, a student will be able to:

- Define the intersection and the sum of subspaces of a vector space.
- State the dimension formula.
- List the conditions for a vector space to be qualified as direct sum of its subspaces.

6.1. Definition

A **vector space** (also called a linear space) is a collection of objects called vectors, which may be added together and multiplied by numbers.

Activity 6.1

Consider $V=\{(2x,0,5x),x\in\mathbb{R}\}$

1. What would be the value of x so that $(0,0,0)\in V$?
2. Let $\vec{u}=(2a,0,5a)$, $\vec{v}=(2b,0,5b)\in V$, $a,b\in\mathbb{R}$. Show that for any real number α,β , $\alpha\vec{u}+\beta\vec{v}\in V$.
3. From 1) and 2) indicate whether V is a sub-vector space.

A subset V of $\mathbb{R}^n$ is called a **sub-vector space**, or just a **subspace** of $\mathbb{R}^n$ if it has the following properties:

- The null vector belongs to V .
- V is closed under vector addition, i.e if $\vec{u},\vec{v}\in V$ then $\vec{u}+\vec{v}\in V$.
- V is closed under scalar multiplication, i.e if $\alpha\in\mathbb{R}$, $\vec{u}\in V$, $\alpha\vec{u}\in V$.

Generally,

If $(\mathbb{R},F,+)$ is a subspace of $(\mathbb{R},E,+)$, then

- $F\subset E$
- $\vec{0}\in F$
- $\vec{u},\vec{v}\in F,\alpha,\beta\in\mathbb{R};\quad \alpha\vec{u}+\beta\vec{v}\in F$

Notice

Let V be a vector space. Then,

- V is a subspace of V
- Also, $\{0\}$ is a subspace of V
- V and $\{0\}$ are called the **trivial** (or **improper**) **subspaces** of V. Other subspaces are called **proper subspaces.**

Example 6.1

Consider $A=\{(-3x,0,4x),x\in\mathbb{R}\}$, show that $(\mathbb{R},A,+)$ is a sub-vector space of $\mathbb{R}^3$.

Solution

- $A\in\mathbb{R}^3$
- If we take $x=0$, we see that $(0,0,0)\in A$
- Consider $k=(-3k,0,4k)$, $t=(-3t,0,4t)\in A$, $\alpha,\beta\in IR$

$$\begin{aligned}\Rightarrow \alpha k+\beta t&=\alpha(-3k,0,4k)+\beta(-3t,0,4t)\\&=(-3\alpha k,0,4\alpha k)+(-3\beta t,0,4\beta t)\\&=(-3\alpha k-3\beta t,0,4\alpha k+4\beta t)\\&=(-3(\alpha k+\beta t),0,4(\alpha k+\beta t))=(-3y,0,4y)\qquad \textit{for}\ \ y=\alpha k+\beta t\end{aligned}$$

Then $\alpha k+\beta t\in A$; therefore, A is a subspace of $\mathbb{R}^3$.

Example 6.2

Consider the subset $U=\{(x_1,y_1):x_1y_1=0\}$. Is U a subspace of $\mathbb{R}^2$?

Solution

We can check that for $x_1=0$ and $y_1=0$; we have $\vec{0}\in U$.

However, notice that $(1,0)\in U$ and $(0,1)\in U$. Yet

$(1,0)+(0,1)=(1,1)\notin U$. Therefore, U is not a subspace of $\mathbb{R}^2$.

Application activity 6.1

1. Is a) $S=\left\{\begin{bmatrix}x\\y\end{bmatrix}:x\geq 0,y\geq 0\right\}$ a subspace of $\mathbb{R}^2$? Why?

   b) Is $S=\left\{\begin{bmatrix}a\\b\\0\end{bmatrix}:a,b\in\mathbb{R}\right\}$ a subspace of $\mathbb{R}^3$? Why?

2. Which of the following are subspaces of $\mathbb{R}^3$? For each which is not subspace, explain why. Otherwise, write down two distinct non-zero vectors $\vec{v}_1, \vec{v}_2$, of the subspace or show that the subspace is just the zero vector.

   a) $\{(x, y, z) \in \mathbb{R}^3 : z = 0\}$

   b) $\{(x, y, z) \in \mathbb{R}^3 : x - 2y + 3z = 0\}$

   c) $\{(x, y, z) \in \mathbb{R}^3 : x - 2y + 3z + 4 = 0\}$

3. From 1) and 2) indicate whether v is a sub vector space.

6.2. Intersection and sum of two vector spaces

6.2.1. Intersection of subspaces

Activity 6.2

Let $H = \{(x, y, z) : 2x - y + 3z = 0\}$ and

$K = \{(x, y, z) : x + y + z = 0\}$ be the subspaces of $\mathbb{R}^3$.

The intersection $H \cap K$ is found by solving the system

$$\begin{cases} 2x - y + 3z = 0 \\ x + y + z = 0 \end{cases}$$

Solve this system and hence deduce $H \cap K$.

Verify whether $H \cap K$ is a subspace of $\mathbb{R}^3$.

From activity 6.2, let U and W be subspaces of a vector space V. The intersection of U and W, written $U \cap W$, consists of all vectors $\vec{u}$ where $\vec{u} \in U$ and $\vec{u} \in W$.

Theorem 6.1

Any intersection of subspaces of a vector space V is a subspace of V.

Example 6.3

Describe the intersection $H \cap K$ if $H = \{(a + b, 3a - b, 2a + b) : a, b \in \mathbb{R}\}$ and $K = \{(3c, c, 2c) : c \in \mathbb{R}\}$.

Solution

The intersection consists of vectors that can be written in the form $(a+b,3a-b,2a+b)$ and in the form $(3c,c,2c)$ for some $a,b,c\in\mathbb{R}$. Thus, they are obtained by solving the following system:

$$\begin{cases}a+b=3c\\3a-b=c\\2a+b=2c\end{cases}\Rightarrow\begin{cases}a+b-3c=0\\3a-b-c=0\\2a+b-2c=0\end{cases}\Rightarrow\begin{cases}a+b-3(3a-b)=0\\c=3a-b\\2a+b-2(3a-b)=0\end{cases}$$

By solving this system, we obtain $\{(0,0,0)\}$. Hence $H\cap K=\{(0,0,0)\}$

Example 6.4

Consider $F=\{(x,0,z),x,z\in\mathbb{R}\}$ and $G=\{(x,y,0),x,y\in\mathbb{R}\}$. Find $F\cap G$

Solution

We need to solve the system

$$\begin{cases}x=x\\0=y\\z=0\end{cases}$$

Then

$$F\cap G=\{(x,0,0),x\in\mathbb{R}\}$$
$$=\{x(1,0,0),x\in\mathbb{R}\}$$

which is generated by $(1,0,0)$

Application activity 6.2

1. Let V be the vector space of 2 by 2 matrices over $\mathbb{R}$. Let U consists of those matrices in V whose second row is zero, and let W consist of those matrices in V whose second column is zero. Find the intersection $U\cap W$.

2. Let $H=\{function\ f\ on\,\mathbb{R}:f(2)=0\}$ and $K=\{function\ f\ on\,\mathbb{R}:f(1)=0\}$. Find $H\cap K$

3. Take U to be the $x-axis$ and V to be the $y-axis$, both subspaces of $\mathbb{R}^2$. Find their intersection.

4. Let $U_1=\{(x,y,0):x,y\in\mathbb{R}\}$ and $U_2=\{(0,y,y):y\in\mathbb{R}\}$. Find their intersection.

5. Let $U_1 = \{(x,y,z): x+y-z=0, 2x-3y+z=0\}$ and $U_2 = \{(x,y,z): x+y-z=0, x-2y+3z=0\}$ be subspaces of $\mathbb{R}^3$. Find their intersection.

Dimension of intersection of subspaces

Activity 6.3

Let $U = \{(x,y,0): x,y \in \mathbb{R}\}$ and $W = \{(x,y,z): x,y,z \in \mathbb{R}\}$ be subspaces of $\mathbb{R}^3$.

Find the a) intersection $U \cap W$.

b) Find the basis of $U \cap W$ and hence deduce the dimension of $U \cap W$.

Hint:

Recall from Senior Five that, a set S of linearly independent vectors which is a spanning set of vector space V is called a basis and number of vectors in S is the dimension of the vector space V.

We recall from Senior Five, that

A finite set S of vectors in a vector space V is called a basis for V provided that:

- The vectors in S are linearly independent.
- The vector in S span V (or V is a generating set of V).

The unique number of vectors in each basis for V is called the **dimension** of V and is denoted by $\dim(V)$.

Example 6.5

Let $U = \{(x,y,0): x,y \in \mathbb{R}\}$ and $V = \{(0,x,y): x,y \in \mathbb{R}\}$ be subspaces of $\mathbb{R}^3$. Find the dimension of their intersection.

Solution

First we need the intersection:

$$\begin{cases} x = 0 \\ y = x \\ 0 = y \end{cases}$$

Be careful!

At this point for this problem, extra care is needed for correct deductions to be made.

Here, first entry is fixed to zero, second entry cannot be fixed to zero even if $y = x$.

The reason is that we can write $V = \{(0, x, y) : x, y \in \mathbb{R}\}$ as $V = \{(0, a, b) : a, b \in \mathbb{R}\}$ and the system becomes

$$\begin{cases} x = 0 \\ y = a \\ 0 = b \end{cases}$$

So, only first and third entries are fixed to zero.

Then, $U \cap V = \{(0, y, 0) : y \in \mathbb{R}\}$

But, $(0, y, 0) = y(0,1,0)$. So, the basis is $\{(0,1,0)\}$ and hence $\dim(U \cap V) = 1$.

Example 6.6

Let $H = \{(a+b, 3a-b, 2a+b) : a, b \in \mathbb{R}\}$ and $K = \{(x, y, z) : x + y - z = 0, x - 2y + 3z = 0\}$ be subspaces of $\mathbb{R}^3$.

Find the dimension of their intersection.

Solution

First, we need the intersection:

The intersection consists of the vectors of the form $(a+b, 3a-b, 2a+b)$ satisfying the system

$$\begin{cases} x + y - z = 0 \\ x - 2y + 3z = 0 \end{cases}$$

Substituting the form $(a+b,3a-b,2a+b)$ into the system gives

$$\begin{cases} a+b+3a-b-2b-b=0 \\ a+b-6a+2b+6a+3b=0 \end{cases} \text{ or } \begin{cases} 2a-b=0 \\ a+6b=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \end{cases}$$

Then, $(H \cap K)=0$ and $\dim(H \cap K)=0$

Example 6.7

Let $U=\{(a,b,c):a-2b+c=0\}$ and $V=\{(a,b,c):a+b-c=0\}$ be subspaces of $\mathbb{R}^3$. Find the dimension of their intersection.

Solution

First, we need the intersection by solving the system

$$\begin{cases} a-2b+c=0 \\ a+b-c=0 \end{cases} \Rightarrow \begin{cases} a-2b+c=0 \\ \underline{a+b-c=0} \end{cases}$$

$$2a-b=0 \Rightarrow b=2a$$

$$a-2b+c=0 \Rightarrow a-4a+c=0 \Rightarrow c=3a$$

Then, $U \cap V=\{(a,2a,3a):a \in \mathbb{R}\}$

But, $(a,2a,3a)=a(1,2,3)$. So, the basis is $\{(1,2,3)\}$ and hence $\dim(U \cap V)=1$

Application activity 6.3

1. Let U and W be the following subspaces of $\mathbb{R}^4$:
   $U=\{(a,b,c,d):b+c+d=0\}$, $W=\{(a,b,c,d):a+b=0,c=2d\}$
   .
   Find the dimension of $U \cap W$.
2. Let $U=\{(a,0,c):a,c \in \mathbb{R}\}$ and $W=\{(0,b,c):b,c \in \mathbb{R}\}$ be subspaces of $\mathbb{R}^3$. Find the $\dim(U \cap W)$.
3. Let $H=\{(a+b,3a-b,b):a,b \in \mathbb{R}\}$ and
   $K=\{(a+2b,3a+b,-b):a,b \in \mathbb{R}\}$ be subspaces of $\mathbb{R}^3$. Find the dimension of their intersection.

6.2.2. Sum of subspaces

Activity 6.4

Let $U=\{(a,0,c):a,c\in\mathbb{R}\}$ and $W=\{(0,b,b):b\in\mathbb{R}\}$ be subspaces of $\mathbb{R}^3$. Find the sum $U+W$. Verify whether $U+W$ is a subspace of $\mathbb{R}^3$.

Let U and W be subspaces of a vector space V. The sum of U and W, written $U+W$, consists of all sums $x+y$ where $x\in U$ and $y\in W$ or $U+W=\{x+y:x\in U \text{ and } y\in W\}$.

Theorem 6.2

- The sum $U+W$ of the subspaces U and V is also a subspace of V.
- W_1 and W_2 are subspaces of V, then W_1+W_2 is the smallest subspace that contains both W_1 and W_2.

Example 6.8

Let $U=\{(a+b,3a-b,2a+b):a,b\in\mathbb{R}\}$ and $V=\{(3a,a,2a):a\in\mathbb{R}\}$ be subspaces of $\mathbb{R}^3$. Find their sum.

Solution

Since there is a in U and in V, we take another unknown c. The sum is

$$\begin{aligned}U+V&=\{(a+b,3a-b,2a+b)+(3c,c,2c):a,b,c\in\mathbb{R}\}\\&=\{(a+b+3c,3a-b+c,2a+b+2c):a,b,c\in\mathbb{R}\}\end{aligned}$$

Example 6.9

Let $U=\{(a,b,c):a-2b+c=0,a=3c\}$ and $V=\{(a,b,c):a+b-c=0,a=b\}$ be subspaces of $\mathbb{R}^3$.

Find their sum.

Solution

First, we solve simultaneously two equations for each subspace:

For $U=\{(a,b,c): a-2b+c=0, a=3c\}$

$$\begin{cases} a-2b+c=0 \\ a=3c \end{cases}$$

$\Rightarrow 3c-2b+c=0 \Rightarrow -2b=-4c \Rightarrow b=2c$

Then, we can write $U=\{(3c,2c,c): c\in\mathbb{R}\}$

For $V=\{(a,\mathrm{b},\mathrm{c}): a+b-c=0, a=b\}$

$$\begin{cases} a+b-c=0 \\ \quad a=b \end{cases}$$

$\Rightarrow b+b-c=0$
$\Rightarrow 2b=c$

Then, we can write $V=\{(b,b,2b): b\in\mathbb{R}\}$

Now, the sum is

$$\begin{aligned} U+V &= \{(3c,2c,c)+(b,b,2\mathrm{b}): b,c\in\mathbb{R}\} \\ &= \{(3c+b, 2c+b, c+2b): b,c\in\mathbb{R}\} \end{aligned}$$

Application activity 6.4

1. Suppose that U and W are subspaces of a vector space V, and that $\{u_i\}$ generates U and $\{w_j\}$ generates W. Show that $\{u_i, w_j\}$, i.e, $\{u_i\}\cup\{w_j\}$, generates $U+V$.

2. Let V be the vector space of 2 by 2 matrices over $\mathbb{R}$. Let U consists of those matrices in V whose second row is zero, and let W consist of those matrices in V whose second column is zero. Find the sum $U+W$.

3. Let $U_1=\{(x,y,z): x+y-z=0, 2x-3y+z=0\}$ and $U_2=\{(x,y,z): x+y-z=0, x-2y+3z=0\}$ be subspaces of $\mathbb{R}^3$. Find their sum.

Dimension of sum of subspaces

Activity 6.5

Let $U=\{(a,0,0): a\in\mathbb{R}\}$ and $W=\{(0,b,0): b\in\mathbb{R}\}$ be subspaces of $\mathbb{R}^3$.

1. Find the sum $U+W$.
2. Find the basis of $U+W$ and hence deduce the dimension of $U+W$.

Hint:

Recall (from Senior Five) that, a set S of linearly independent vectors which is a spanning set of vector space V is called a basis and the number of vectors in S is the dimension of the vector space V.

We recall from Senior Five, that;

A finite set S of vectors in a vector space V is called a basis for V provided that:

- The vectors in S are linearly independent.
- The vector in S span V (or S is a generating set of V).

The unique number of vectors in each basis for V is called the dimension of V and is denoted by $\dim(V)$.

Example 6.10

Let $U=\{(a,2a-b,3a-b): a,b\in\mathbb{R}\}$ and $V=\{(a,3a,2a): a\in\mathbb{R}\}$ be subspaces of $\mathbb{R}^3$. Find their sum and dimension of $U+V$.

Solution

Since there is a in U and in V, we take another unknown c. The sum is

$$\begin{aligned}U+V&=\{(a,2a-b,3a-b)+(c,3c,2c): a,b,c\in\mathbb{R}\}\\&=\{(a+c,2a-b+3c,3a-b+2c): a,b,c\in\mathbb{R}\}\end{aligned}$$

Now,

$$\begin{aligned}(a+c,2a-b+3c,3a-b+2c)&=(a,2a,3a)+(0,-b,-b)+(c,3c,2c)\\&=a(1,2,3)+b(0,-1,-1)+c(1,3,2)\end{aligned}$$

We need to check if $(1,2,3),(0,-1,-1)$ and $(1,3,2)$ are linearly independent.

We solve

$$\alpha(1,2,3)+\beta(0,-1,-1)+\gamma(1,3,2)=(0,0,0)$$

$$\begin{cases}\alpha+\gamma=0\\ 2\alpha-\beta+3\gamma=0\\ 3\alpha-\beta+2\gamma=0\end{cases}$$

The only solution is $\{(0,0,0)\}$. So, vectors $(1,2,3),(0,-1,-1)$ and $(1,3,2)$ are linearly independent.

Then, the basis for $U+V$ is $S=\{(1,2,3),(0,-1,-1),(1,3,2)\}$ and hence $\dim(U+V)=3$.

Example 6.11

$\forall(a,b)\in\mathbb{R}^2$, we have $(a,b)=a(1,0)+b(0,1)$. Thus, $\{(1,0),(0,1)\}$ is the basis of $\mathbb{R}^2$ and $\dim(\mathbb{R}^2)=2$.

Example 6.12

$\forall(a,b,c)\in\mathbb{R}^3$, we have

$$(a,b,c)=(a,0,0)+(0,b,0)+(0,0,c)=a(1,0,0)+b(0,1,0)+c(0,0,1)$$

Thus, $\{(1,0,0),(0,1,0),(0,0,1)\}$ is the basis of $\mathbb{R}^3$ and $\dim(\mathbb{R}^3)=3$.

Application activity 6.5

1. Consider two subspaces $H=\left\{\begin{pmatrix}2a & 0\\ 3a & 0\end{pmatrix}: a\in\mathbb{R}\right\}$ and $K=\left\{\begin{pmatrix}0 & b\\ b & 3b\end{pmatrix}: b\in\mathbb{R}\right\}$. Find the dimension of $H+K$.

2. Let $J=\{(a,b,c): a-2b+c=0, a=3c$ and $V=\{(a,b,c): a+b-c=0, a=c\}$ be subspaces of $\mathbb{R}^3$. Find the dimension of their sum.

3. Let $U_1=\{(x,y,z): x+y-z=0, 2x-3y+z=0\}$ and $U_2=\{(x,y,z): x+y-z=0, x-2y+3z=0\}$ be subspaces of $\mathbb{R}^3$. Find the dimension of their sum.

Grassmann's formula of dimensions

Activity 6.6

Let $F=\{(x,0,z):x,z\in\mathbb{R}\}$ and $G=\{(0,y,z):y,z\in\mathbb{R}\}$.

1. Find a) $\dim(F)$ and $\dim(G)$

   b) $\dim(F)+\dim(G)$

   c) $F\cap G$ and $\dim(F\cap G)$

   d) $\dim(F)+\dim(G)-\dim(F\cap G)$

   e) $F+G$ and $\dim(F+G)$

2. Compare your results from d) and e).

From activity 6.6,

If $(\mathbb{R},F,+)$ and $(\mathbb{R},G,+)$ are two sub-vector spaces of $(\mathbb{R},E,+)$, we have, $\dim(F+G)=\dim(F)+\dim(G)-\dim(F\cap G)$. This formula is called **Grassmann's formula of dimensions**.

Example 6.13

Consider $F=\{(x,0,z),x,z\in\mathbb{R}\}$ and $G=\{(x,y,0),x,y\in\mathbb{R}\}$. Verify Grassmann's formula.

Solution

- **For** F

$$\left.\begin{aligned}(x,0,z)&=(x,0,0)+(0,0,z)\\&=x(1,0,0)+z(0,0,1)\end{aligned}\right\}\Rightarrow\dim(F)=2$$

- **For** G

$$\left.\begin{aligned}(x,y,0)&=(x,0,0)+(0,y,0)\\&=x(1,0,0)+y(0,1,0)\end{aligned}\right\}\Rightarrow\dim(G)=2$$

- $F+G=\{(2x,y,z),x,y,z\in\mathbb{R}\}$

$$\left.\begin{aligned}(2x,y,z)&=(2x,0,0)+(0,y,0)+(0,0,z)\\&=x(2,0,0)+y(0,1,0)+z(0,0,1)\end{aligned}\right\}\Rightarrow \dim(F+G)=3$$

- $F\cap G=\{(x,0,0),x\in\mathbb{R}\}$

$$\left.\begin{aligned}(x,0,0)&=(x,0,0)\\&=x(1,0,0)\end{aligned}\right\}\Rightarrow \dim(F\cap G)=1$$

Then,

$$\begin{aligned}\dim(F+G)=3\\&=\dim(F)+\dim(G)-\dim(F\cap G)\\&=2+2-1=3\quad \text{hence verified.}\end{aligned}$$

Notice

If $\dim(F\cap G)=0$, then $\dim(F+G)=\dim(F)+\dim(G)$. In this case, F and G are said to be **complementary** and the sum $F+G$ is said to be a **direct sum**; and it is denoted by $F\oplus G$. Otherwise, F and G are said to be **supplementary**.

Theorem 6.3

- The vector space V is the direct sum of its subspaces W_1 and W_2 (i.e, $V=W_1\oplus W_2$) if and only if $V=W_1+W_2$ and $W_1\cap W_2=\{0\}$.
- Let W_1 and W_2 be two subspaces of a vector space V over F, and then $V=W_1\oplus W_2 \Leftrightarrow \forall x\in V, \exists!\ x_1\in W_1, \exists!\ x_2\in W_2$ such that $x=x_1+x_2$.

  As, if we suppose $x=x_1+x_2=y_1+y_2$, $x_1,y_1\in W_1, x_2,y_2\in W_2$, then, $x_1-y_1=y_2-x_2$ and $x_1-y_1\in W_1$, $y_2-x_2\in W_2$.

  Therefore, $x_1-y_1=y_2-x_2\in W_1\cap W_2=\{0\}$

  $\Rightarrow x_1=y_1,\ x_2=y_2$.

Example 6.14

Let W_1, W_2, and W_3 denote the *x-axis*, the *y-axis*, and the $z-axis$ respectively. Show that $\mathbb{R}^3$ is uniquely represented as a direct sum of W_1, W_2, and W_3.

Solution

$\mathbb{R}^3 = W_1 \oplus W_2 \oplus W_3$, $W_i \cap W_j = \{0\}, i \neq j$.

$\forall (a,b,c) \in \mathbb{R}^3$, $(a,b,c) = (a,0,0) + (0,b,0) + (0,0,c)$

where $(a,0,0) \in W_1,\ (0,b,0) \in W_2,\ (0,0,c) \in W_3$

Therefore, $\mathbb{R}^3$ is uniquely represented as a direct sum of W_1, W_2, and W_3.

Example 6.15

Let $U = \{(a,b,0) : a,b \in \mathbb{R}\}$ be the $xy-plane$ and let $W = \{(0,0,c) : c \in \mathbb{R}\}$ be the $z-axis$. Show that $\mathbb{R}^3$ is a direct sum of U and W.

Solution

Any vector $(a,b,c) \in \mathbb{R}^3$ can be written as the sum of a vector in U and a vector in V in one and only one way: $(a,b,c) = (a,b,0) + (0,0,c)$.

Accordingly, $\mathbb{R}^3$ is a direct sum of U and W, that is, $\mathbb{R}^3 = U \oplus W$.

Application activity 6.6

1. Given V and W, the sub-vector spaces of $\mathbb{R}^4$ such that $V = \{(a,b,c,d) : b - 2c - d = 0\}$ and $W = \{(a,b,c,d) : a = d, b = 2c\}$. Find the dimension of V, W and $V \cap W$. Deduce $\dim(V + W)$.
2. Let W_1 and W_2 denote the xy and the $\mathbb{R}^3$ planes, respectively. Can $\mathbb{R}^3$ be uniquely represented as a direct sum of W_1 and W_2? Show your working steps.
3. If W_1 and W_2 are the set of all even functions and the set of all odd functions respectively. Is $F = W_1 + W_2$ a direct sum? Show your working steps.
4. Let $F = \{(x,y,0) : x,y \in \mathbb{R}\}$ and $G = \{(0,w,z) : w,z \in \mathbb{R}\}$ Is the sum of F and G a direct sum? Show your working steps.
5. Assume that U and W are distinct subspaces $(U \neq W)$ of a four-dimensional vector space V and $\dim(U) = \dim(W) = 3$. Prove that $\dim(U \cap W) = 2$.

Unit summary

1. Definition

If $(\mathbb{R}, F, +)$ is a subspace of $(\mathbb{R}, E, +)$, then

- $F \subset E$
- $\vec{0} \in F$
- $\vec{u}, \vec{v} \in F, \alpha, \beta \in \mathbb{R}; \quad \alpha\vec{u} + \beta\vec{v} \in F$

2. Intersection and sum of two vector spaces

Let U and W be subspaces of a vector space V. The intersection of U and W, written $U \cap W$, consists of all vectors $\vec{u}$ where $\vec{u} \in U$ and $\vec{u} \in W$.

Any intersection of subspaces of a vector space V is a subspace of V.

If F and G are two subspaces of E, then, the sum of F and G is also a subspace of E. It is denoted as $F + G = \{x + y, x \in F, y \in G\}$

Grassmann's formula of dimensions.

If $(\mathbb{R}, F, +)$ and $(\mathbb{R}, G, +)$ are two sub-vector spaces of $(\mathbb{R}, E, +)$, we have, $\dim(F+G) = \dim(F) + \dim(G) - \dim(F \cap G)$.

Remark

If $\dim(F \cap G) = 0$, then, $\dim(F+G) = \dim(F) + \dim(G)$. In this case, F and G are said to be **complementary** and the sum $F + G$ is said to be a **direct sum**; and it is denoted by $F \oplus G$.

Otherwise, F and G are said to be **supplementary**.

End of unit assessment

1. In Exercise a-d, which one is a subspace of $\mathbb{R}^3$?
   a) The plane $x = y$
   b) The line $(1+t, 2t, 3t)$
   c) The locus $x^2 + y^2 + z^2 = 0$
   d) The locus $x^2 + y^2 - z^2 = 0$
2. Find the dimension of subspaces E and F if

   $$E = \{(x, y) : x + y = 0\}$$

   $$F = \{(x, y, z) : 2x - y + z = 0\}$$
3. For $W = \{(a_1, a_2, a_3, a_4, a_5) \in \mathbb{R}^5 : a_1 + a_3 + a_5 = 0, a_2 = a_4\}$, find $dim(W)$.
4. Suppose U and W are distinct 4-dimensional subspaces of a vector space V of dimension 6. Find the possible dimensions of $U \cap W$.
5. Let U and W be the subspaces of $\mathbb{R}^4$ generated by $\{(1,1,0,-1),(1,2,3,0),(2,3,3,-1)\}$ and $\{(1,2,2,-2),(2,3,2,-3),(1,3,4,-3)\}$ respectively. Find
   a) $\dim(U + W)$ b) $\dim(U \cap W)$
6. Show that the set of all square matrices can be decomposed into the direct sum of the set of the symmetric matrices and that of the skew-symmetric ones.
7. Let $F = \{(x, 0, 0) : x \in \mathbb{R}\}$ and $G = \{(0, y, 0) : x \in \mathbb{R}\}$. Is $W = F + G$ a direct sum? Show your working steps.

Unit 7

Transformation of Matrices

Introductory activity

Given a matrix $A=\begin{pmatrix}3 & 4\\-1 & 2\end{pmatrix}$ and a point P with coordinates (x,y) of the Cartesian plane.

a) Find the coordinate of the point P' which are (x',y') such that

$$\begin{bmatrix}x'\\y'\end{bmatrix}=\begin{pmatrix}3 & 4\\-1 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

b) If f is a transformation by which P' is the image of P, find the image of the point O(0,0) and A (1,2). Present each point and its image in the same Cartesian plan.

c) Is the point and its image the same?

d) Does the matrix have an effect on the position of an object or the object remains in its position? Explain your answer.

Matrices are used in Cryptogram where a message is written according to a secret code. This code uses matrices to encode and decode messages for example when sending money in the telephone.

Objectives

By the end of this unit, I will be able to:

- Define the kernel, the image, the nullity and the rank of a linear transformation.
- State the dimension formula for linear transformations.
- Carry out the elementary row operations on matrices.
- Define and find eigenvalues and eigenvectors of a square matrix.
- Discuss the diagonalisation of square matrices.

Matrices and their inverse are used by programmers for coding or encrypting a message. Matrices are applied in the study of electrical circuits, quantum mechanics and optics. A message is made as a sequence of numbers in a binary format for communication and it follows code theory for solving.

7.1. Kernel and range of a transformation

Activity 7.1

Consider the following relation:

$f:\mathbb{R}^2 \to \mathbb{R}^2$

$f(x,y)=(3x+y+2,3x-y+1)$

Find all vectors (x,y) such that $f(x,y)=(0,0)$.

- The **kernel** of a linear mapping $f:E\to F$ denoted $Ker(f)$ is a subset of E whose image by f is 0-vector of F. i.e, $Ker(f)=\{v\in E: f(v)=0\}$.
- The **nullity** of f denoted $n(f)$ is the dimension of $Ker(f)$. i.e, $n(f)=\dim Ker(f)$.
- The **image** or **range** of a linear mapping $f:E\to F$ is the set of points in F to which points in E are mapped on. i.e, $\operatorname{Im}(f)=\{u\in F: f(v)=u\}, u\in E$ and $\operatorname{Im}(f)$ is a

  vector subspace of F.
- The **rank** of f denoted $rank(f)$ or $r(f)$ is the dimension of image of f.

  i.e, $rank(f)=\dim(\operatorname{Im} f)$.

Notice

A linear transformation f is called **singular** if there exists a non-zero vector whose image is zero vector. Thus, it is **non-singular** if the only zero vector has zero vector as

image, or equivalently, if its kernel consists only of the zero vector: $Ker(f)=\{0\}$.

Theorems

- A linear transformation $f: E \to F$ is one-to-one (1-1) if and only if $Ker(f) = \{0\}$.
- A linear transformation $f: E \to F$ is onto if the range is equal to F.
- Consider the linear transformation $f: E \to F$, the following is true: $\dim[Ker(f)] + \dim[range(f)] = \dim(E)$.
- Consider the linear transformation $f: E \to F$, If $\dim(E) = \dim(F)$, then,
  - a) f is one-to-one.
  - b) f is onto.

In this case, $f: E \to F$ is called an **isomorphism**. And we say that E and F are isomorphic, and we write $E \cong F$.

Example 7.1

Let f be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ defined by $f(\vec{v}) = A\vec{v}$ with $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{pmatrix}$

a) Find a basis for $Ker(f)$.

b) Determine if f is one to one.

c) Find a basis for the range of f.

d) Determine if f is onto.

Solution

Let $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, then $Av = \begin{pmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+3b \\ 2a+6b \\ 3a+9b \end{pmatrix}$

a) To find the kernel of f, we set $\begin{pmatrix} a+3b \\ 2a+6b \\ 3a+9b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then

$$\begin{cases} a+3b=0 \\ 2a+6b=0 \Rightarrow \{a+3b=0 \Rightarrow a=-3b \quad or \quad b=-\dfrac{a}{3} \\ 3a+9b=0 \end{cases}$$

The kernel of *f* is the set of all vectors of the form $\begin{pmatrix} -3b \\ b \end{pmatrix}, \ b \in \mathbb{R}$

To find the basis: $\begin{pmatrix} -3b \\ b \end{pmatrix} = b\begin{pmatrix} -3 \\ 1 \end{pmatrix}$, then the basis is $\{(-3,1)\}$.

Therefore, the basis for $Ker(f)$ is $\{(-3,\ 1)\}$.

b) f is not one to one since the $Ker(f) \neq 0$

c) Range of f has the form

$$\begin{pmatrix} a+3b \\ 2a+6b \\ 3a+9b \end{pmatrix}, \quad a,b \in \mathbb{R}$$

To find basis

$$\begin{pmatrix} a+3b \\ 2a+6b \\ 3a+9b \end{pmatrix} = \begin{pmatrix} a \\ 2a \\ 3a \end{pmatrix} + \begin{pmatrix} 3b \\ 6b \\ 9b \end{pmatrix} = a\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + b\begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

The basis for the range would be $\{(1,2,3),(3,6,9)\}$ but we see that the vector $(3,6,9)$ is a multiple of the vector $(1,2,3)$. This means that the two vectors are linearly dependent. Then, the vector $(3,6,9)$ must be removed. Hence, the basis is $\{(1,2,3)\}$.

d) Since the dimension of the range of A is 1 and the dimension of $\mathbb{R}^3$ is 3, f is not onto.

Example 7.2

Consider the linear mapping

$t:\mathbb{R}^3 \to \mathbb{R}^3$

$$t(x,y,z)=(x+2y-z,y+z,x+y-2z)$$

Find:

a) $Ker(t)$ b) Range of t and $rank(t)$

Solution

a) $Ker(t)$:

We have $\begin{cases} x+2y-z=0 & (1) \\ y+z=0 & (2) \\ x+y-2z=0 & (3) \end{cases}$

From (2), $y=-z \quad (4)$

(4) in (1) and (3) gives $\begin{cases} x-2z-z=0 \\ x-z-2z=0 \end{cases}$ or $x=3z$.

Then, the vector (x,y,z) becomes $(3z,-z,z)$

Hence, kernel of t is $Ker(t)=\{(3z,-z,z),\ z\in IR\}$.

Basis and dimension:

$(3z,-z,z)=z(3,-1,1)$

The basis is $\{(3,-1,1)\}$ and $n(t)=1$

b) Range of t

The range of t is $\{(x+2y-z,y+z,x+y-2z),x,y,z\in\mathbb{R}\}$

Basis and dimension:

$$\begin{pmatrix} x+2y-z \\ y+z \\ x+y-2z \end{pmatrix}=\begin{pmatrix} x \\ 0 \\ x \end{pmatrix}+\begin{pmatrix} 2y \\ y \\ y \end{pmatrix}+\begin{pmatrix} -z \\ z \\ -2z \end{pmatrix}=x\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}+y\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}+z\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

Next, we check if the vector, $\{(1,0,1),(2,1,1),(-1,1,-2)\}$ are linearly independent. It can be seen that $(2,1,1)=3(1,0,1)+(-1,1,-2)$, means that the three

vectors are linearly dependent. Then, we remove the second vector, $(2,1,1)$, and the basis of image of t is $\{(1,0,1),(-1,1,-2)\}$.

Hence, $rank(t)=2$.

Application activity 7.7

1. Let $F:V \to U$ be the projection mapping into the $x-y$ plane: $F(x,y,z)=(x,y,0)$. Find:

   a) $\operatorname{Im} F$ b) $KerF$

2. Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear mapping defined by $'(x,y,z)=(x+2y-z,y+z,x+y-2z$. Find a basis and the dimension of the:

   a) Image U of T b) Kernel W of T

3. Let $F:\mathbb{R}^4 \to \mathbb{R}^3$ be the linear mapping defined by $'(x,y,s,t)=(x-y+s+t,x+2s-t,x+y+3s-3t$.
   Find a basis and the dimension of the:

   a) Image U of F b) Kernel W of F

7.2. Elementary row/column operations

Activity 7.2

Consider matrix $A=\begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}$

Perform the following operations on matrix A:

1. New $row\ 2 \to row\ 2 + row\ 1$
2. New $row\ 3 \to row\ 3 - 3row\ 1$, use matrix obtained in 1).
3. New $column\ 2 \leftrightarrow column\ 3$, use matrix obtained in 2).
4. New $column\ 3 \to column\ 3 - \frac{1}{3} column\ 2$, use matrix obtained in 3).

Elementary matrix operations are of **three** kinds:

a) Interchanging two lines.
b) Multiplying each element in a line by a non-zero number.
c) Multiplying a line by a non-zero number and adding the result to another line.

Note that here the term **line** is used to mean either a row or a column of the matrix.

When these operations are performed on rows, they are called **elementary row operations**; and when they are performed on columns, they are called **elementary column operations**.

Notation

In many references, you will encounter a compact notation to describe elementary operations. That notation is shown below:

Operation description		Notation
Row operations		
1. Interchange row i and j	$\rightarrow$	$r_i \leftrightarrow r_j$
2. Multiply row i by $s \neq 0$	$\rightarrow$	$new\ r_i \rightarrow sr_i$
3. Add s times row i to row j	$\rightarrow$	$new\ r_j \rightarrow r_j + sr_i$
Column operations		
1. Interchange column i and j	$\rightarrow$	$c_i \leftrightarrow c_j$
2. Multiply column i by $s \neq 0$	$\rightarrow$	$new\ c_i \rightarrow sc_i$
3. Add s times column i to column j	$\rightarrow$	$new\ c_j \rightarrow c_j + sc_i$

Example 7.3

$$A = \begin{pmatrix} 1 & 2 & 3 & -4 \\ 2 & -3 & 1 & 2 \\ 5 & 0 & -2 & 7 \end{pmatrix}$$

- The operation $r_2 \leftrightarrow r_3$ performed to A gives

$$B = \begin{pmatrix} 1 & 2 & 3 & -4 \\ 5 & 0 & -2 & 7 \\ 2 & -3 & 1 & 2 \end{pmatrix}$$

- The operation $c_3 \rightarrow -2c_3$ performed to B gives

$$C = \begin{pmatrix} 1 & 2 & -6 & -4 \\ 5 & 0 & 4 & 7 \\ 2 & -3 & -2 & 2 \end{pmatrix}$$

- The operation $r_2 \to r_2 + 4r_1$ performed to C gives

$$D = \begin{pmatrix} 1 & 2 & -6 & -4 \\ 9 & 8 & -20 & -9 \\ 2 & -3 & -2 & 2 \end{pmatrix}$$

Then, matrix D is obtained from matrix A by a sequence of elementary operations.

Definitions

- Two matrices are said to be **row equivalent** (or **column equivalent**) if one can be changed to the other by a sequence of elementary row (or column) operations.

  The concept of equivalence should not be confused with that of similarity, which is only defined for **square matrices**, and it is much more restrictive (similar matrices are certainly equivalent, but equivalent square matrices need not be similar).

- Two matrices A and B are said to be **similar** if $B = P^{-1}AP$ for some invertible matrix P. If A and B are similar, we write $A \sim B$. Similar matrices represent the same linear transformation under two different bases.

Example 7.4

Show that the matrices $A = \begin{pmatrix} 4 & -2 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 \\ 1 & 2 \end{pmatrix}$ are similar given invertible matrix $P = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$.

Solution

$A = \begin{pmatrix} 4 & -2 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -2 \\ 1 & 2 \end{pmatrix}$ are similar because for

$P = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ we have

$$B = P^{-1}AP = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 4 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 2 \end{pmatrix}$$

Since $B = P^{-1}AP = \begin{pmatrix} 3 & -2 \\ 1 & 2 \end{pmatrix}$, then A and B are similar.

Application activity 7.8

1. Consider matrix $A=\begin{pmatrix}1 & 2 & -3 & 0\\ 2 & 4 & -2 & 2\\ 3 & 6 & -4 & 3\end{pmatrix}$.

   Perform the following operations on matrix A:

   a) $r_2 \to -2r_1 + r_2$ b) $r_3 \to -3r_1 + r_3$

   c) $r_3 \to -5r_2 + 4r_3$

2. Consider matrix $A=\begin{pmatrix}2 & 3 & 2 & 4 & 6\\ 0 & 0 & 3 & 2 & 5\\ 0 & 0 & 0 & 0 & 2\end{pmatrix}$.

   Perform the following operations on matrix A:

   a) $r_1 \to -4r_2 + 3r_1$ b) $r_1 \to r_3 + r_1$ c) $r_2 \to -5r_3 + 2r_2$

7.3. Diagonalisation of matrices

7.3.1. Eigenvalues and eigenvectors

Activity 7.3

Consider matrix $A=\begin{pmatrix}4 & 2\\ 3 & -1\end{pmatrix}$

1. Find the determinant, $\det(A-\lambda I)$ where I is identity matrix of order 2 and $\lambda \in \mathbb{R}$.
2. Equate the determinant obtained in 1) to zero. Hence, find the value(s) of λ by solving equation formed.
3. Using the value(s) of λ obtained in 2), find the vector(s) $\vec{u}=\begin{pmatrix}u_1\\ u_2\end{pmatrix}$ if $(A-\lambda I)\vec{u}=\vec{0}$.

Definitions:

Definition 1: Given any vector space, E and any linear map $f: E \to E$, a scalar $\lambda \in K$ is called an **eigenvalue**, or **proper value**, or **characteristic value** of f if there is some non-zero vector $\vec{u} \in E$ such that $f(\vec{u}) = \lambda \vec{u}$.

Equivalently, λ is an eigenvalue of f if $Ker(f - \lambda I)$ is non-trivial.

i.e, $Ker(f - \lambda I) \neq 0$; where I is identity matrix.

Definition 2: A vector $\vec{u} \in E$ is called an **eigenvector**, or **proper vector**, or **characteristic vector** of f if there is some $\lambda \in K$ such that $f(\vec{u}) = \lambda \vec{u}$ and $\vec{u} \neq \vec{0}$.

The scalar λ is then an eigenvalue, and we say that $\vec{u}$ is an **eigenvector associated with** $\lambda \in K$.

Definition 3: Given any eigenvalue $\lambda \in K$, the non-trivial subspace $Ker(f - \lambda I)$ consists of all the eigenvectors associated with λ together with the zero vector; this subspace is denoted by $E_\lambda(f)$, or even E_λ, and is called the **eigenspace associated with** λ, or **proper subspace associated with** λ.

Note that distinct eigenvectors may correspond to the same eigenvalue, but distinct eigenvalues correspond to disjoint set of eigenvectors.

Definition 4: The eigenvalues of f are the roots (in K) of the polynomial equation: $\det(f - \lambda I) = 0$.

This polynomial is a polynomial associated with f and is called **characteristic polynomial**.

For any square matrix A, the polynomial $\det(A - \lambda I)$ is its characteristic polynomial.

The homogeneous system $(f - \lambda I)\vec{u} = \vec{0}$ gives the eigenvector $\vec{u}$ associated with eigenvalue λ.

For any square matrix A, the solution of homogeneous system $(A - \lambda I)\vec{u} = \vec{0}$, for the value of λ, is an eigenvector associated with λ.

Note that to have an eigenvector, it must be a **non-trivial solution** of the system.

Example 7.5

Find eigenvalues and eigenvectors if $A=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$

Solution

Eigenvalues:

$$\det(A-\lambda I)=0$$

$$\det\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]=0$$

$$\begin{vmatrix}-\lambda & 1\\ 1 & -\lambda\end{vmatrix}=0\Rightarrow \lambda^2-1=0\Rightarrow \lambda=1 \ or \ \lambda=-1$$

The eigenvalues are -1 and 1.

Eigenvectors

For $\lambda=-1$,

$$\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]\vec{u}=\vec{0}$$

$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{cases}u_1+u_2=0\\ u_1+u_2=0\end{cases}\quad \Rightarrow u_2=-u_1 \quad or \quad u_1=-u_2$$

It doesn't matter the substitution we will make (i.e, we can take $u_2=-u_1 \quad or \quad u_1=-u_2$).

Taking $u_2=-u_1$, eigenvectors associated with $\lambda=-1$ have the form $u_1\begin{pmatrix}1\\ -1\end{pmatrix}$, $u_1\in\mathbb{R}_0$. We can take $\vec{u}=\begin{pmatrix}1\\ -1\end{pmatrix}$.

For $\lambda=1$,

$$\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}-\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]\vec{v}=\vec{0}$$

$$\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{cases}-v_1+v_2=0\\ v_1-v_2=0\end{cases}\Rightarrow v_2=v_1$$

Each vector of the form $v_1\begin{pmatrix}1\\ 1\end{pmatrix}, v_1\in\mathbb{R}_0$ is an eigenvector associated with $\lambda=1$. We can take $\vec{v}=\begin{pmatrix}1\\ 1\end{pmatrix}$.

Example 7.6

Find the eigenvalues and associated eigenvectors for matrix $B=\begin{pmatrix}3 & -1\\ 1 & 1\end{pmatrix}$ as matrix over $\mathbb{R}$.

Solution

We have

$$\det\left[\begin{pmatrix}3 & -1\\ 1 & 1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]=0 \qquad \Leftrightarrow (\lambda-2)^2=0$$
$$\Rightarrow \lambda=2$$

Hence, only 2 is an eigenvalue.

Now,

$$\left[\begin{pmatrix}3 & -1\\ 1 & 1\end{pmatrix}-2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]\begin{pmatrix}u_1\\ u_2\end{pmatrix}=0$$

$$\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \quad \Leftrightarrow \begin{cases}u_1-u_2=0\\ u_1-u_2=0\end{cases}\Rightarrow u_1-u_2=0\Rightarrow u_1=u_2$$

Each vector of the form $u_1\begin{pmatrix}1\\ 1\end{pmatrix}, u_1\in\mathbb{R}_0$ is an eigenvector associated with $\lambda=2$. We can take $\vec{u}=\begin{pmatrix}1\\ 1\end{pmatrix}$.

Example 7.7

Find the eigenvalues and associated eigenvectors for the following matrix over $\mathbb{R}$;

$$C=\begin{pmatrix}1 & -1\\ 2 & -1\end{pmatrix}$$

Solution

We have

$$\det\left[\begin{pmatrix}1 & -1\\ 2 & -1\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right]=0 \Leftrightarrow \lambda^2+1=0$$

Since $\lambda^2+1=0$ has no solution in $\mathbb{R}$, C has no eigenvalue as matrix over $\mathbb{R}$.

Notice

- The characteristic polynomial $\det(A-\lambda I)$ is sometimes written as $\det(\lambda I-A)$.
- Eigenvalue relationships:

  If $\lambda_1,\ldots\ldots,\lambda_n$ are the eigenvalues of matrix A, then $tr(A)=\sum_i \lambda_i$ and $\det(A)=\prod_i \lambda_i$ ($\sum \equiv \text{summation}$, $\prod \equiv \text{product}$).

Example 7.8

From Example 7.6, $B=\begin{pmatrix}3 & -1\\ 1 & 1\end{pmatrix}$,

$tr(B)=3+1=4$ and $|B|=3\times1-1\times(-1)=4$.

But we have seen that the eigenvalue is 2 which is a double root of the characteristic polynomial, then $\lambda_1=\lambda_2=2$.

Thus, $tr(B)=2+2=4$ and $\det(B)=2\times2=4$.

Some important properties of Eigenvalues

- Any square matrix A and its transpose A^t have the same eigenvalues.

- For triangular matrix $A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, the

  eigenvalues are the elements of diagonal entries $a_{11}, a_{22}, \ldots a_{nn}$, as

$$\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda)\ldots(a_{nn} - \lambda).$$

- If $\lambda = 0$ is one of Eigenvalues of matrix A, thus A is singular i.e. A^{-1} does not exist.

- **Cayley-Hamilton** states that "**Every square matrix satisfies its own characteristic equation**".

  That is to say, if $|A - \lambda \mathrm{I}| = \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_n$ is characteristic polynomial of matrix $A = A_{n \times n}$, then, matrix equation $X^n + a_1 X^{n-1} + a_2 X^{n-2} + \cdots + a_n I = 0$ is satisfied by $X = A$ i.e. $A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I = 0$.

Example 7.9

Determine the characteristic equation of the matrix $A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$ and verify that it is satisfied by A and hence obtain A^{-1}.

Solution

Characteristic equation is $|A - \lambda I| = 0$

$$\Leftrightarrow \begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (1-\lambda)\left[(1-\lambda)(-1-\lambda) - 3\right] - 2(-1-\lambda-1) - 2(3-1+\lambda) = 0$$

$$\Leftrightarrow (1-\lambda)(\lambda^2 - 1 - 3) + 2(\lambda+2) - 2(2+\lambda) = 0$$

$$\Leftrightarrow (1-\lambda)(\lambda+2)(\lambda-2) + 2(\lambda+2) - 2(2+\lambda) = 0$$

$$\Leftrightarrow (\lambda+2)\left[(1-\lambda)(\lambda-2)+2-2\right]=0$$

$$\Leftrightarrow (1-\lambda)(\lambda+2)(\lambda-2)=0$$

$$\Leftrightarrow (1-\lambda)\left(\lambda^2-4\right)=0$$

$$\Leftrightarrow \lambda^3-\lambda^2-4\lambda+4=0$$

Thus, Characteristic equation is $\lambda^3-\lambda^2-4\lambda+4=0$.

By Cayley-Hamilton theorem, $A^3-A^2-4A+4I=0$.

Let us verify whether $A^3-A^2-4A+4I=0$.

$$A^2=\begin{pmatrix}1 & 2 & -2\\ 1 & 1 & 1\\ 1 & 3 & -1\end{pmatrix}\begin{pmatrix}1 & 2 & -2\\ 1 & 1 & 1\\ 1 & 3 & -1\end{pmatrix}=\begin{pmatrix}1 & -2 & 2\\ 3 & 6 & -2\\ 3 & 2 & 2\end{pmatrix}$$

$$A^3=\begin{pmatrix}1 & -2 & 2\\ 3 & 6 & -2\\ 3 & 2 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & -2\\ 1 & 1 & 1\\ 1 & 3 & -1\end{pmatrix}=\begin{pmatrix}1 & 6 & -6\\ 7 & 6 & 2\\ 7 & 14 & -6\end{pmatrix}$$

$$A^3-A^2-4A+4I=\begin{pmatrix}1 & 6 & -6\\ 7 & 6 & 2\\ 7 & 14 & -6\end{pmatrix}-\begin{pmatrix}1 & -2 & 2\\ 3 & 6 & -2\\ 3 & 2 & 2\end{pmatrix}-4\begin{pmatrix}1 & 2 & -2\\ 1 & 1 & 1\\ 1 & 3 & -1\end{pmatrix}+4\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$=\begin{pmatrix}1-1-4+4 & 6+2-8 & -6-2+8\\ 7-3-4 & 6-6-4+4 & 2+2-4\\ 7-3-4 & 14-2-12 & -6-2+4+4\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Thus, it is verified that the characteristic equation is satisfied by A.

Inverse of matrix *A*

From $A^3-A^2-4A+4I=0$, multiplying on both sides by A^{-1} yields

$A^2-A-4I+4A^{-1}=0$ or $4A^{-1}=-A^2+A+4I$

$$\Leftrightarrow 4A^{-1}=-\begin{pmatrix}1 & -2 & 2\\ 3 & 6 & -2\\ 3 & 2 & 2\end{pmatrix}+\begin{pmatrix}1 & 2 & -2\\ 1 & 1 & 1\\ 1 & 3 & -1\end{pmatrix}+4\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\Leftrightarrow 4A^{-1} = \begin{pmatrix} -1+1+4 & 2+2+0 & -2-2+0 \\ -3+1+0 & -6+1+4 & 2+1+0 \\ 3+1 & -2+3+0 & -2-1+4 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{4}\begin{pmatrix} 4 & 4 & -4 \\ -2 & -1 & 3 \\ -2 & 1 & 1 \end{pmatrix}$$

Application activity 7.9

1. Determine eigenvalues and eigenvectors for each of the following matrices:

   a) $A = \begin{pmatrix} 5 & 6 \\ 3 & -2 \end{pmatrix}$ b) $B = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$ c) $C = \begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}$

2. For each matrix, find all eigenvalues and a basis for each eigen space.

   a) $A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix}$ b) $B = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}$ c) $C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

3. Prove that a matrix A and its transpose A^t have the same eigenvalues.

4. Show that for triangular matrix $A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$,

   the eigenvalues are the elements of diagonal entries $a_{11}, a_{22}, \ldots a_{nn}$.

5. Use Cayley-Hamilton theorem to find the inverse of matrix

   $A = \begin{pmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{pmatrix}$.

6. If $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$, then express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ in terms of A.

7.3.2. Diagonalisation

Activity 7.4

Consider matrix $A = \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}$

1. Find the eigenvalues and their associated eigenvectors (they must be linearly independent).
2. Form matrix P whose columns are elements of eigenvectors obtained in 1).
3. Find the inverse of matrix P obtained in 2).
4. Find matrix D by relation $D = P^{-1}AP$. What can you say about matrix D?

Diagonalising a square matrix A is to find a diagonal matrix D such that for an invertible matrix P:

$A = PDP^{-1}$ or $D = P^{-1}AP$

When this happens, we say that A is **diagonalisable**.

Every symmetric matrix can be diagonalised, however, not every matrix can be diagonalised.

To diagonalise matrix A, we perform the following steps:

1. Find the eigenvalues.
2. If there is a non-real eigenvalue, the matrix cannot be diagonalised.
3. If all eigenvalues are real, find their associated eigenvectors (they must be linearly independent).
4. If the number of eigenvectors is not equal to the order of matrix A, then this matrix cannot be diagonalised.
5. If the number of eigenvectors is equal to the order of matrix A, form matrix P whose columns are elements of eigenvectors.
6. Find the inverse of P.
7. Find D, diagonal matrix of A by relation $D = P^{-1}AP$.

Theorem

A $n \times n$ matrix is diagonalisable if and only if it has n linearly independent eigenvectors.

Example 7.10

Diagonalise the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Solution

Eigenvalues and eigenvectors:

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0 \Rightarrow (1+\lambda)^2(2-\lambda) = 0 \quad \Rightarrow \begin{cases} \lambda_1 = -1 \quad (double\ root) \\ \lambda_2 = 2 \end{cases}$$

For $\lambda = -1$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{cases} u_1 + u_2 + u_3 = 0 \\ u_1 + u_2 + u_3 = 0 \Rightarrow u_1 + u_2 + u_3 = 0 \\ u_1 + u_2 + u_3 = 0 \end{cases}$$

Since we have used an eigenvalue which is a double root, we would get two eigenvectors and they must be linearly independent.

As $u_1 + u_2 + u_3 = 0$ is a plane, it is possible to find two linearly independent vectors on this plane.

Thus, we can take;

$$\vec{u} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

For $\lambda = 2$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{cases} -2w_1 + w_2 + w_3 = 0 \\ w_1 - 2w_2 + w_3 = 0 \\ w_1 + w_2 - 2w_3 = 0 \end{cases} \Rightarrow w_1 = w_2 = w_3$$

Thus, $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Now, the eigenvectors are $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and are linearly independent. The number of eigenvectors is equal to the order of the given matrix.

$$P = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

After calculation,

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \text{ in the basis } \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Example 7.11

Diagonalise the matrix

$$A = \begin{pmatrix} 5 & -3 \\ 3 & -1 \end{pmatrix}$$

Solution

Eigenvalues and eigenvectors

$$\begin{vmatrix} 5-\lambda & -3 \\ 3 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda - 2)^2 = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = 2$$

$$\begin{pmatrix} 2 & -3 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 3u_1 - 3u_2 = 0 \\ 3u_1 - 3u_2 = 0 \end{cases}$$

$$\Rightarrow u_1 = u_2$$

Eigenvectors associated with $\lambda_1 = \lambda_2 = 2$ have the form $u_1\begin{pmatrix}1\\1\end{pmatrix}$.

As we have two equal eigenvalues, we would have two independent eigenvectors. But we see that it is not possible because all eigenvectors are spanned by $\begin{pmatrix}1\\1\end{pmatrix}$.

Therefore, the given matrix cannot be diagonalised.

Remarks

- To form matrix P, we start with the eigenvector of our choice. Then, it doesn't matter the vector we start with.
- Note that in the diagonal matrix of A, the diagonal entries are just the eigenvalues corresponding to the eigenvectors.

Application activity 7.10

1. Diagonalise each of the following matrices:

   a) $A = \begin{pmatrix}7 & 3\\3 & -1\end{pmatrix}$ b) $B = \begin{pmatrix}2 & -1\\-2 & 3\end{pmatrix}$ c) $C = \begin{pmatrix}5 & 4\\4 & -1\end{pmatrix}$

   d) $D = \begin{pmatrix}5 & 6\\-2 & -2\end{pmatrix}$ e) $E = \begin{pmatrix}11 & -8 & 4\\-8 & -1 & -2\\4 & -2 & -4\end{pmatrix}$

2. Consider matrix $A = \begin{pmatrix}3 & -1 & 1\\7 & -5 & 1\\6 & -6 & 2\end{pmatrix}$

   a) Find all eigenvalues of A.

   b) Find a maximum set S of linearly independent eigenvectors of A.

   c) Is A diagonalisable? If yes, find matrix P such that $D = P^{-1}AP$ is diagonal.

7.4. Applications

7.4.1. Echelon Matrix

Activity 7.5

Consider the following matrix $A = \begin{pmatrix} 8 & 3 \\ 1 & 2 \end{pmatrix}$

Use elementary row operations to transform this matrix such that:

1. The first non-zero element in each row is 1 and is in a column to the right of the other in the previous row.
2. Rows with all zero elements, if any, are below rows having a non-zero element.
3. The first non-zero element in each row is the only non-zero entry in its column.

A matrix is in **row echelon form** (ref) when it satisfies the following conditions:

- The first non-zero element in each row, called the **leading entry**, is 1.
- Each leading entry is in a column to the right of the leading entry in the previous row.
- Rows with all zero elements, if any, are below rows having a non-zero element.

A matrix is in **reduced row echelon form** (rref) when it satisfies the following conditions:

- The matrix is in row echelon form (i.e., it satisfies the three conditions listed above).
- The leading entry in each row is the only non-zero entry in its column.

A matrix in echelon form is called an **echelon matrix**. Matrices A and B below are some examples of echelon matrices.

$$A = \begin{pmatrix} 1 & 2 & 3 & 3 & 4 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Matrix A is in row echelon form, and matrix B is in reduced row echelon form.

How to transform a matrix into its Echelon Forms

Any matrix can be transformed into its echelon forms, using a series of elementary row operations. Here is how.

a) Pivot the matrix

- Identify the pivot; the first non-zero entry in the first column of the matrix.
- If the pivot identified is not in the first row and first column, interchange rows by moving the pivot row to the first row.
- Multiply each element in the pivot row by the inverse of the pivot, so the pivot equals 1.
- Add or subtract multiples of the pivot row to each of the lower rows, so every element in the pivot column of the lower rows equals 0.

b) To get the matrix in row echelon form,

- Repeat the procedures above, ignoring previous pivot rows.
- Continue until there are no more pivots to be processed.

c) To get the matrix in reduced row echelon form, process non-zero entries above each pivot.

- Identify the last row having a pivot equal to 1, and let this be the pivot row.
- Add multiples of the pivot row to each of the upper rows, until every element above the pivot equals 0.
- Moving up the matrix, repeat this process for each row.

The matrix in reduced row echelon form obtained from matrix A is called its **row canonical form.**

Example 7.12

Transform the following matrix in its echelon form

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 7 & 8 \end{pmatrix}$$

Solution

$$A=\begin{pmatrix}0&1&2\\1&2&1\\2&7&8\end{pmatrix}\xrightarrow{r_1\leftrightarrow r_2}B=\begin{pmatrix}1&2&1\\0&1&2\\2&7&8\end{pmatrix}$$

$$\xrightarrow{r_3\to r_3-2r_1}C=\begin{pmatrix}1&2&1\\0&1&2\\0&3&6\end{pmatrix}\quad\xrightarrow{r_3\to r_3-3r_2}D=\begin{pmatrix}1&2&1\\0&1&2\\0&0&0\end{pmatrix}$$

$$\xrightarrow{r_1\to r_1-2r_2}F=\begin{pmatrix}1&0&-3\\0&1&2\\0&0&0\end{pmatrix}$$

Matrix D is the row echelon form of matrix A.

Matrix F is the reduced row echelon form of matrix A. F is the row canonical form of A.

Application activity 7.11

Transform the following matrices in their echelon form and row reduced form:

1. $\begin{pmatrix}0&2&8&-7\\2&-2&4&0\\-3&4&-2&-5\end{pmatrix}$

2. $\begin{pmatrix}1&-2&3&9\\-1&3&0&-4\\2&-5&5&17\end{pmatrix}$

3. $\begin{pmatrix}2&-2&4&-2\\2&1&10&7\\-4&4&-8&4\\4&-1&14&6\end{pmatrix}$

4. $\begin{pmatrix}3&-2&4&7\\2&1&0&-3\\2&8&-8&2\end{pmatrix}$

5. $\begin{pmatrix}3&-1&2&4&1\\2&1&3&-1&2\\1&2&3&-2&3\end{pmatrix}$

6. $\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}$

7.4.2. Matrix inverse

Activity 7.6

Consider matrix $A=\begin{pmatrix}1&1&1\\0&1&2\\1&2&4\end{pmatrix}$ written in the form

$M=\left(A\,|\,I\right)$ that is $M=\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&1&2&0&1&0\\1&2&4&0&0&1\end{array}\right)$;

1. Perform elementary row operations on matrix A such that the matrix A in M becomes a unit matrix.
2. Multiply the new matrix A obtained in 1) by matrix A and give your observation.

The elementary operations can be used to find the inverse of matrix A. The method used here is called the Gaussian elimination method.

Steps to follow

For a square matrix A of order n, to compute the inverse of A, denoted as A^{-1}, we follow the steps below:

1. Construct a matrix of type $M=\left(A\,|\,I\right)$, that is to say, A is in the left half of M and the identity matrix I is on the right.
2. Using the Gaussian elimination method, transform the left half, A, to an identity matrix and the matrix that results in the right side will be the inverse of matrix A.

Example 7.13

Using elementary row operations on the matrix A, determine its matrix inverse where

$$A=\begin{pmatrix}1&1&0\\1&0&1\\0&1&0\end{pmatrix}$$

Solution

Place the identity matrix of order 3 to the right of matrix A.

$$M = \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Perform elementary row operations on matrix A so that the matrix A in M becomes a unit matrix.

new $r_2 \to r_2 - r_1$ $$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

new $r_3 \to r_3 + r_2$ $$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right)$$

new $r_2 \to r_2 - r_3$ $$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right)$$

new $r_1 \to r_1 + r_2$ $$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right)$$

new $r_2 \to (-1) \cdot r_2$ $$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right)$$

Therefore, the inverse matrix is $A^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$

Application activity 7.12

Use Gaussian elimination method to find the inverse of the following matrices:

1. $\begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 3 & 1 \\ -1 & 2 & 0 \\ 2 & 11 & 3 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 2 & 1 \\ -2 & 1 & 8 \\ 1 & -2 & -7 \end{pmatrix}$

7.4.3. Rank of matrix

Activity 7.7

Consider matrix $A = \begin{pmatrix} 4 & -6 & 0 \\ -6 & 0 & 1 \\ 0 & 9 & -1 \\ 0 & 1 & 4 \end{pmatrix}$

1. Transform matrix A in its row echelon form using elementary row operations.
2. How many non-zero rows are there in the matrix obtained in 1)?

The rank of matrix is the number of linearly independent rows or columns. Using this definition, the Gaussian elimination method is used to find the rank.

To compute the rank of a matrix, remember two key points:

a) The rank does not change under elementary row operations.
b) The rank of a row-echelon matrix is easy to acquire.

Recall that we can convert a given matrix into row echelon form using elementary row operations.

A line can be discarded if:

- All the coefficients are zeros.
- There are two equal lines.

- A line is proportional to another.
- A line is a linear combination of others.

Example 7.14

Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & -1 & 3 & -2 \\ 2 & 1 & 0 & 1 & 1 \\ 2 & 4 & -2 & 6 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 5 & 4 & -1 & 5 & 0 \end{pmatrix}$$

Solution

$r_3 = 2 \cdot r_1$

r_4 is zero

$r_5 = 2r_2 + r_1$

The remaining two rows are linearly independent and are non-zero.

Then, $r(A) = 2$.

In general, eliminate the maximum possible number of lines, and the rank is the number of non-zero rows.

Example 7.15

Find the rank of the matrix

$$A = \begin{pmatrix} 1 & -4 & 2 & -1 \\ 3 & -12 & 6 & -3 \\ 2 & -1 & 0 & 1 \\ 0 & 1 & 3 & -1 \end{pmatrix}$$

Solution

Transform matrix A to echelon matrix:

new $r_2 \to r_2 - 3r_1$ $$\begin{pmatrix} 1 & -4 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 1 \\ 0 & 1 & 3 & -1 \end{pmatrix}$$

new $r_3 \to r_3 - 2r_1$ $$\begin{pmatrix} 1 & -4 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 7 & -4 & 3 \\ 0 & 1 & 3 & -1 \end{pmatrix}$$

new $r_3 \to r_3 - 7r_4$ $$\begin{pmatrix} 1 & -4 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -25 & 10 \\ 0 & 1 & 3 & -1 \end{pmatrix}$$

$r_2 \leftrightarrow r_4$ $$\begin{pmatrix} 1 & -4 & 2 & -1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & -25 & 10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that there are 3 non-zero rows.

Then, $r(A) = 3$.

Example 7.16

Calculate the rank of the following matrix:

$$\begin{pmatrix} 2 & -1 & 0 & 7 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & 7 & 7 \end{pmatrix}$$

Solution

Transform the matrix to echelon matrix:

new $r_1 \to r_1 - 2r_2$ $$\begin{pmatrix} 0 & -1 & -2 & 1 \\ 1 & 0 & 1 & 3 \\ 3 & 2 & 7 & 7 \end{pmatrix}$$

new $r_3 \to r_3 - 3r_2$ $\begin{pmatrix} 0 & -1 & -2 & 1 \\ 1 & 0 & 1 & 3 \\ 0 & 2 & 4 & -2 \end{pmatrix}$

new $r_3 \to r_3 + 2r_1$ $\begin{pmatrix} 0 & -1 & -2 & 1 \\ 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

new $r_1 \leftrightarrow r_2$ $\begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Then, $r(A) = 2$ since there are two non-zero rows.

Application activity 7.13

Use elementary row operations to find the rank of the following matrices:

1. $\begin{pmatrix} 0 & 16 & 8 & 4 \\ 2 & 4 & 8 & 16 \\ 16 & 8 & 4 & 2 \\ 4 & 8 & 16 & 2 \end{pmatrix}$

2. $\begin{pmatrix} 3 & 0 & 1 & 2 \\ 6 & 1 & 0 & 0 \\ 12 & 1 & 2 & 4 \\ 9 & 0 & 1 & 2 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & 2 & 2 & 1 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 2 & 1 & -1 \\ 9 & 5 & 2 & 2 \\ 7 & 1 & 0 & 4 \end{pmatrix}$

7.4.4. Solving system of linear equations

Activity 7.8

Consider the following system of linear equations

$$\begin{cases} x + y + z = 6 \\ 2x + y - z = 1 \\ 3x + 2y + z = 10 \end{cases}$$

1. Determine the matrix A such that $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ 10 \end{pmatrix}$

2. Make A in 1), the larger matrix 3×4 (called augmented matrix) where the fourth column is formed by the independent terms of the given system.

3. Transform the matrix obtained in 2) to its row echelon form.

4. Use the result obtained in 3) to find the value of x, y and z.

Consider the following system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = c_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = c_m \end{cases} \quad (1)$$

Where $x_1, x_2, ..., x_n$ are unknowns;

a_{ij} and c_i are real constants.

The Gauss elimination method is used to transform a system of equations into an equivalent system, that is, in row echelon form.

For easy calculation, transform the system into a matrix and place the coefficients of the variables and the independent terms into the matrix as follows:

$$(A:C) = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} & :c_1 \\ a_{21} & a_{22} & ... & a_{2n} & :c_2 \\ \vdots & \vdots & ... & \vdots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} & :c_m \end{pmatrix}$$

where;

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} & :c_1 \\ a_{21} & a_{22} & ... & a_{2n} & :c_2 \\ \vdots & \vdots & ... & \vdots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} & :c_m \end{pmatrix} \text{ and } C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

The matrix $(A:C)$ is called **augmented matrix**.

Remarks

- If $rank(A) \neq rank(A:C)$, the system is said to be inconsistent and there is no solution.
- If $rank(A) = rank(A:C) = r$, the system is said to be consistent and there is solution.
  - » If $r = n$, as there are n unknowns, then the system has a unique solution.
  - » If $r < n$, the system has infinite solutions. (It is **undetermined system**).

Example 7.17

Solve the following system

$$\begin{cases} x+y+z=-3 \\ 3x+y-2z=-2 \\ 2x+4y+7z=7 \end{cases}$$

Solution

The augmented matrix is $\begin{pmatrix} 1 & 1 & 1 & : & -3 \\ 3 & 1 & -2 & : & -2 \\ 2 & 4 & 7 & : & 7 \end{pmatrix}$

Performing the row reductions, we have;

$$\xrightarrow{\begin{cases} r_2 \to r_2 - 3r_1 \\ r_3 \to r_3 - 2r_1 \end{cases}} \begin{pmatrix} 1 & 1 & 1 & : & -3 \\ 0 & -2 & -5 & : & 7 \\ 0 & 2 & 5 & : & 13 \end{pmatrix} \xrightarrow{r_3 \to r_3 + r_2} \begin{pmatrix} 1 & 1 & 1 & : & -3 \\ 0 & -2 & -5 & : & 7 \\ 0 & 0 & 0 & : & 20 \end{pmatrix}$$

We see that

$$rank(A) = 2,\ rank(A:C) = 3 \Rightarrow rank(A) \neq rank(A:C)$$

Then, the system is inconsistent. Therefore, there is no solution. (This is because for the third row, we have $0z = 20$ which is not possible).

Example 7.18

Solve the following system

$$\begin{cases} x+y+z=4 \\ 2x+y-z=1 \\ x-y+2z=2 \end{cases}$$

Solution

$$\begin{pmatrix} 1 & 1 & 1 & : & 4 \\ 2 & 1 & -1 & : & 1 \\ 1 & -1 & 2 & : & 2 \end{pmatrix}$$

$$\xrightarrow{\begin{cases} r_2 \to r_2-2r_1 \\ r_3 \to r_3-r_1 \end{cases}} \begin{pmatrix} 1 & 1 & 1 & : & 4 \\ 0 & -1 & -3 & : & -7 \\ 0 & -2 & 1 & : & -2 \end{pmatrix} \xrightarrow{r_3 \to r_3-2r_2} \begin{pmatrix} 1 & 1 & 1 & : & 4 \\ 0 & -1 & -3 & : & -7 \\ 0 & 0 & 7 & : & 12 \end{pmatrix}$$

We see that $rank(A) = rank(A:C) = 3$, then, the system has solution.

The reduced system is

$$\begin{cases} x+y+z=4 \\ -y-3z=-7 \\ 7z=12 \end{cases} \Rightarrow \begin{cases} x=\dfrac{3}{7} \\ y=\dfrac{13}{7} \\ z=\dfrac{12}{7} \end{cases}$$

Therefore, $S = \left\{\left(\dfrac{3}{7}, \dfrac{13}{7}, \dfrac{12}{7}\right)\right\}$

Notice

It is also possible to transform the system in the form where the elements above and below the leading diagonal of matrix A become zeros. The system is now reduced to the simplest system.

Example 7.19

Solve the following system

$$\begin{cases} 2x+y+z=10 \\ 3x+2y+3z=18 \\ x+4y+9z=16 \end{cases}$$

Solution

$$\begin{pmatrix} 2 & 1 & 1 & : 10 \\ 3 & 2 & 3 & : 18 \\ 1 & 4 & 9 & : 16 \end{pmatrix} \xrightarrow{\begin{cases} r_2 \to r_2 - \frac{3}{12} r_1 \\ r_3 \to r_3 - \frac{1}{2} r_1 \end{cases}} \begin{pmatrix} 3 & 1 & 1 & : 10 \\ 0 & \frac{1}{2} & \frac{3}{2} & : 3 \\ 0 & \frac{7}{2} & \frac{17}{2} & : 11 \end{pmatrix}$$

$$\xrightarrow{r_3 \to r_3 - 7r_2} \begin{pmatrix} 3 & 1 & 1 & :10 \\ 0 & \frac{1}{2} & \frac{3}{2} & :3 \\ 0 & 0 & -2 & :-10 \end{pmatrix} \xrightarrow{r_1 \to r_1 - 2r_2} \begin{pmatrix} 2 & 0 & -2 & :4 \\ 0 & \frac{1}{2} & \frac{3}{2} & : 3 \\ 0 & 0 & -2 & :-10 \end{pmatrix}$$

$$\xrightarrow{\begin{cases} r_1 \to r_1 - r_3 \\ r_2 \to r_2 + \frac{3}{4} r_3 \end{cases}} \begin{pmatrix} 2 & 0 & 0 & :14 \\ 0 & \frac{1}{2} & 0 & : \frac{-9}{2} \\ 0 & 0 & -2 & :-10 \end{pmatrix}$$

Now, the reduced system is

$$\begin{cases} 2x = 14 \\ \frac{1}{2} y = -\frac{9}{2} \\ -2z = -10 \end{cases} \Rightarrow x = 7, y = -9 z = 5$$

Then,

$$S = \{(7, -9, 5)\}$$

Application activity 7.14

Use Gaussian elimination method to solve the following systems:

1. $\begin{cases} 2y + z = -8 \\ x - 2y - 3z = 0 \\ -x + y + 2z = 3 \end{cases}$

2. $\begin{cases} x - 2y - 6z = 12 \\ 2x + 4y + 12z = -17 \\ x - 4y - 12z = 22 \end{cases}$

3. $\begin{cases} x + 2y + 3z = 9 \\ 2x - 2z = -2 \\ 3x + 2y + z = 7 \end{cases}$

4. $\begin{cases} x + 2y + 3z = 9 \\ 3x + 2y + z = 7 \end{cases}$

7.4.5. Power of matrix

Activity 7.9

Let A be a diagonalisable matrix.

From equality $A = PDP^{-1}$, where D is a diagonal matrix, P, an invertible matrix, compute

1. A^2 2. A^3 3. A^4 4. A^5

Deduce the general rule for computing A^n

Hint: $A^2 = AA = PDP^{-1}PDP^{-1}$ and $A^n = A^{n-1}A$

From Activity 7.9, one can deduce the following:

a) The power of matrix A is given by $A^n = PD^nP^{-1}$ for an invertible matrix P whose columns are elements of eigenvectors of matrix A.

Where,

$$D^n = \begin{pmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k^n \end{pmatrix}$$

λ_k are eigenvalues.

b) The inverse of matrix A is given by $A^{-1} = PD^{-1}P^{-1}$.

In fact, since $A \cdot A^{-1} = I$, we have

$$A \cdot \left(PD^{-1}P^{-1}\right) = \left(PDP^{-1}\right) \cdot \left(PD^{-1}P^{-1}\right) = PDP^{-1}PD^{-1}P^{-1} = PDID^{-1}P^{-1}$$

$$= PDD^{-1}P^{-1} = PIP^{-1} = PP^{-1} = I$$

Therefore, $A^{-1} = PD^{-1}P^{-1}$.

Example 7.20

Let $A = \begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}$. Find the non-singular matrix P and the diagonal matrix D such that $D = P^{-1}AP$ and hence find A^n; n is any positive integer.

Solution

We need to find the eigenvalues and eigenvectors of A first. The characteristic equation of A is

$$\det(A-\lambda I)=\begin{vmatrix}-4-\lambda & -6\\ 3 & 5-\lambda\end{vmatrix}=(\lambda+1)(\lambda-2)=0 \Rightarrow \lambda=-1 \text{ or } 2$$

For $\lambda=2$, $\vec{u}=\begin{pmatrix}-1\\1\end{pmatrix}$, for $\lambda=-1$, $\vec{v}=\begin{pmatrix}-2\\1\end{pmatrix}$

Let $P=\begin{pmatrix}-1 & -2\\ 1 & 1\end{pmatrix}$ then, $D=\begin{pmatrix}2 & 0\\ 0 & -1\end{pmatrix}$

To find A^n,

$$D^n=\begin{pmatrix}2^n & 0\\ 0 & (-1)^n\end{pmatrix}$$

Now,

$$PD^nP^{-1}=PP^{-1}A^nPP^{-1}=A^n=\begin{pmatrix}-1 & -2\\ 1 & 1\end{pmatrix}\begin{pmatrix}2^n & 0\\ 0 & (-1)^n\end{pmatrix}\begin{pmatrix}-1 & -2\\ 1 & 1\end{pmatrix}^{-1}$$

$$=\begin{pmatrix}-\left[2^n+2\cdot(-1)^{n+1}\right] & -\left[2^{n+1}+2\cdot(-1)^{n+1}\right]\\ 2^n+(-1)^{n+1} & 2^{n+1}+(-1)^{n+1}\end{pmatrix}$$

Application activity 7.15

For each of the following matrices, find a non-singular matrix P and a diagonal matrix D such that $A=PDP^{-1}$ and hence evaluate the given power.

1. $A=\begin{pmatrix}4 & -12\\ -12 & 11\end{pmatrix}$, A^3
2. $A=\begin{pmatrix}-5 & 8\\ -4 & 7\end{pmatrix}$, A^5
3. $A=\begin{pmatrix}1 & 2\\ 0 & -1\end{pmatrix}$, A^{20}
4. $A=\begin{pmatrix}2 & 0 & 3\\ 0 & 3 & 0\\ 0 & 0 & 3\end{pmatrix}$, A^5

Unit summary

1. Kernel and range

- The **kernel** of a linear mapping $f: E \to F$ denoted $Ker(f)$ is a subset of E whose image f is 0-vector of F .i.e, $Ker(f) = \{v \in E : f(v) = 0\}$
.

  A linear transformation f is called singular if there exists a non-zero vector whose image is zero vector. Thus, it is non-singular if the only zero vector has zero vector as image, or equivalently, if its kernel consists only of the zero vector: $Ker(f) = \{0\}$.

  A linear transformation $f: E \to F$ is one to one if and only if $Ker(f) = \{0\}$.

- The **nullity** of f denoted $n(f)$ is the dimension of $Ker(f)$. i.e, $n(f) = \dim Ker(f)$.
- The **image** or range of a linear mapping $f: E \to F$ is the set of points in F to which points in E are mapped on. i.e, $\operatorname{Im} f = \{u \in F : f(v) = u\}, \ v \in E$.

  A linear transformation $f: E \to F$ is onto if the range is equal to F .
- The **rank** of f denoted $rank(f)$ or $r(f)$ is the dimension of image of f.

  i.e, $rank(f) = \dim(\operatorname{Im} f)$.

2. Elementary row/column operations

When these operations are performed on rows, they are called **elementary row operations**; and when they are performed on columns, they are called **elementary column operations**.

Operation description		Notation
Row operations		
1. Interchange row i and j	$\to$	$r_i \leftrightarrow r_j$
2. Multiply row i by $s \neq 0$	$\to$	$r_i \to sr_i$
3. Add s times row i to row j	$\to$	$r_j \to r_j + sr_i$
Column operations		
1. Interchange column i and j	$\to$	$c_i \leftrightarrow c_j$
2. Multiply column i by $s \neq 0$	$\to$	$c_i \to sc_i$
3. Add s times column i to column j	$\to$	$c_j \to c_j + sc_i$

Two matrices are said to be **row equivalent** (or **column equivalent**) if one can be changed to the other by a sequence of elementary row (or column) operations.

Two matrices A and B are said to be **similar** if $B = P^{-1}AP$ for some invertible matrix P.

3. Diagonalisation of matrices

a) Eigenvalues and eigenvectors

The eigenvalues of f are the roots (in K) of the polynomial: $\det(f - \lambda I)$. This polynomial is a polynomial associated with f and is called characteristic polynomial. For any square matrix A, the polynomial $\det(A - \lambda I)$ is its characteristic polynomial. The homogeneous system $(f - \lambda I)\vec{u} = \vec{0}$ gives the eigenvector $\vec{u}$ associated with eigenvalue λ.

Cayley-Hamilton states that "*Every square matrix satisfies its own characteristic equation*".

a) Diagonalisation

To diagonalise matrix A, we perform the following steps:

(i) Find the eigenvalues.

(ii) If there is a non-real eigenvalue, the matrix cannot be diagonalised.

(iii) If all eigenvalues are real, find their associated eigenvectors (they must be linearly independent).

(iv) If the number of eigenvectors is not equal to the order of matrix A, then this matrix cannot be diagonalised.

(v) If the number of eigenvectors is equal to the order of matrix A, form matrix P whose columns are elements of eigenvectors.

(vi) Find the inverse of P.

(vii) Find D, diagonal matrix of A by relation $D = P^{-1}AP$.

4. Applications

a) Echelon matrix

A matrix is in **row echelon form** (ref) when it satisfies the following conditions:

(i) The first non-zero element in each row, called the leading entry, is 1.

(ii) Each leading entry is in a column to the right of the leading entry in the previous row.

(iii) Rows with all zero elements, if any, are below rows having a non-zero element.

A matrix is in **reduced row echelon form** (rref) when it satisfies the following conditions:

(iv) The matrix is in row echelon form (i.e., it satisfies the three conditions listed above).

(v) The leading entry in each row is the only non-zero entry in its column.

b) Matrix inverse

A is a square matrix of order n. To calculate the inverse of A, denoted as A^{-1}, follow these steps:

(i) Construct a matrix of type $M = (A \mid I)$, that is to say, A is in the left half of M and the identity matrix I is on the right.

(ii) Using the Gaussian elimination method, transform the left half, A, to the identity matrix, located to the right, and the matrix that results in the right side will be the inverse of matrix A.

c) Rank of matrix

The rank of matrix is the number of linearly independent rows or columns. Using this definition, the Gaussian elimination method is used to find the rank.

A line can be discarded if:

» All the coefficients are zeros.

» There are two equal lines.

» A line is proportional to another.

» A line is a linear combination of others.

In general, eliminate the maximum possible number of lines, and the rank is the number of non-zero rows.

d) Solving system of linear equations

Consider the following system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = c_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = c_m \end{cases}$$

The Gauss elimination method is to transform a system of equations into an equivalent system that is in triangular form.

To facilitate the calculation, transform the system into a matrix and place the coefficients of the variables and the independent terms into the matrix as follows:

$$(A:C)=\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & :c_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & :c_2 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & :c_m \end{pmatrix}$$

Where;

The matrix $(A:C)$ is called augmented matrix.

Remarks

- If $rank(A) \neq rank(A:C)$, the system is said to be inconsistent and there is no solution.
- If $rank(A) = rank(A:C) = r$, the system is said to be consistent and there is solution.
  - » If $r = n$, as there are n unknowns, then the system has a unique solution.
  - » If $r < n$, the system has infinite solutions. (It is undetermined system).

e) Power of matrix

The power of matrix A is given by $A^n = PD^nP^{-1}$ for an invertible matrix P whose columns are elements of eigenvectors of matrix A.

Where,

$$D^n = \begin{pmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k^n \end{pmatrix}$$

λ_k are eigenvalues

End of unit assessment

1. In question a-e, find the characteristic polynomial:

 a) $A=\begin{pmatrix}7 & -3\\ 5 & -2\end{pmatrix}$ b) $B=\begin{pmatrix}2 & 5\\ 4 & 1\end{pmatrix}$

 c) $C=\begin{pmatrix}3 & -2\\ 9 & -3\end{pmatrix}$ d) $D=\begin{pmatrix}1 & 6 & -2\\ -3 & 2 & 0\\ 0 & 3 & -4\end{pmatrix}$

 e) $E=\begin{pmatrix}1 & 2 & 3\\ 3 & 0 & 4\\ 6 & 4 & 5\end{pmatrix}$

2. Let $A=\begin{pmatrix}3 & -4\\ 2 & -6\end{pmatrix}$

 a) Find all eigenvalues and corresponding eigenvectors.

 b) Find matrices P and D such that P is non-singular and $D=P^{-1}AP$ is diagonal.

3. Let $B=\begin{pmatrix}2 & 2\\ 1 & 3\end{pmatrix}$

 a) Find all eigenvalues and corresponding eigenvectors.

 b) Find matrices P and D such that P is non-singular and $D=P^{-1}AP$ is diagonal.

 c) Find A^6 and $f(A)$, where $f(t)=t^4-3t^3-6t^2+7t+3$

 d) Find a real cube root of B, that is, a matrix B such that $B^3=A$ and B has real eigenvalues.

4. Consider matrix $A=\begin{pmatrix}4 & 1 & -1\\ 2 & 5 & -2\\ 1 & 1 & 2\end{pmatrix}$

 a) Find all eigenvalues of A.

 b) Find a maximum set S of linearly independent eigenvectors of A.

 c) Find diagonal matrix for A.

5. Given linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z)=(2x+y-2z, 2x+3y-4z, x+y-z)$. Find all eigenvalues of matrix representative of A relative to the canonical basis and diagonalise it.

6. Show that a matrix A and its transpose A^t have the same characteristic polynomial.

7. In question a-d, find the row echelon form of each matrix.

a) $A=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 1 \end{pmatrix}$ b) $B=\begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix}$

c) $C=\begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$ d) $D=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}$

8. In question a-d, compute the rank of each matrix.

a) $A=\begin{pmatrix} 1 & 2 & 0 & 5 \\ 2 & 3 & 1 & 4 \\ -1 & -1 & -1 & 1 \end{pmatrix}$ b) $B=\begin{pmatrix} 1 & 2 & 1 \\ -1 & 3 & 4 \\ 2 & -1 & -3 \end{pmatrix}$

c) $C=\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ -2 & 1 & 4 \end{pmatrix}$ d) $D=\begin{pmatrix} 1 & 3 \\ 2 & -1 \\ -1 & -3 \end{pmatrix}$

9. In question a-b, use Gaussian elimination method to solve the following systems

a) $\begin{cases} 3x-2y+z=-6 \\ 4x-3y+3z=7 \\ 2x+y-z=-9 \end{cases}$ b) $\begin{cases} 3x-2y+z=4 \\ x+3y-z=-3 \\ 4x-10y+4z=10 \end{cases}$

10. Using Cayley-Hamilton theorem, find the inverse of the following matrices:

a) $A=\begin{pmatrix} -3 & 5 & 1 \\ 2 & 10 & 1 \\ 1 & 8 & 1 \end{pmatrix}$ b) $B=\begin{pmatrix} 2 & -2 & 0 \\ 1 & 3 & 4 \\ 3 & 1 & 4 \end{pmatrix}$

c) $C = \begin{pmatrix} 0 & 1 & 0 \\ 2 & -2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

11. Find the characteristic equation of the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$.

Verify Cayley-Hamilton theorem and hence evaluate the matrix expression $-5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A$.

Unit 8 Conics

Introductory activity

The curves that can be obtained by intersecting a double cone with a plane are called **conics** or **conic sections,** the most important of which are circles, ellipse, parabolas and hyperbolas.

Now, lets consider this situation. Suppose you are a gardener, and you have just planted a lot of flowers that you want to water. The flower bed is 450 cm wide and 450 cm long. You are using a circular sprinkler system, and the water reaches 180 cm out from the centre. The sprinkler is located, from the bottom left corner of the bed, 210 cm up, and 180 cm over. If the flower bed was a graph with the bottom left corner being the origin, what would the equation of the circle be?

Today, properties of conic sections are used in the constructions of telescopes, radar antennas, and navigational systems, and in determination of satellite orbits.

Consider the parabolic antenna below

Figure 1.1. *Parabolic antenna*

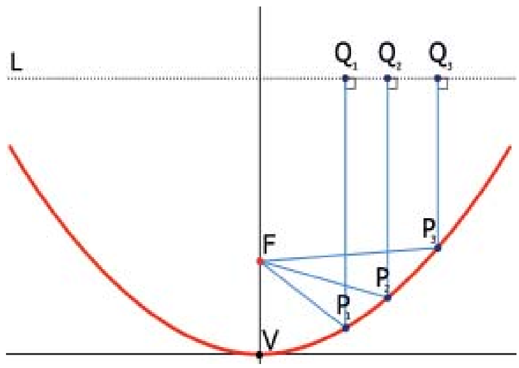


Figure 1.2. *Reflection of all waves to one point*

A **parabolic antenna** is an antenna that uses a parabolic reflector, a curved surface with the cross-sectional shape of a parabola, to direct the radio waves. Figure 8.1 represents parabolic antenna and figure 8.2 shows how parabolic antenna helps to reflect all waves to one point called focal point *F*. The most common form is shaped like a dish. Its main advantage is that it has high directivity. But, how can we find the equation of this dish? You can also think about motion of planets. What can you say about motion of planets around the sun? How can you find the equation of their orbits around the sun?

Objectives

By the end of this unit, a student will be able to:

- Define geometrically a conic as the intersection of a plane and a cone and classify conics from the position of the intersecting plane.
- Express, in Cartesian form, the standard equation of a parabola, an ellipse and a hyperbola.
- Convert Cartesian coordinates into polar coordinates and vice versa.
- Find the polar equation of a conic, a straight line and a circle.
- Use translation or rotation to reduce a general equation of a conic.

8.1. Generalities on conoc sections

Activity 8.1

Consider a double cone in the figure below.

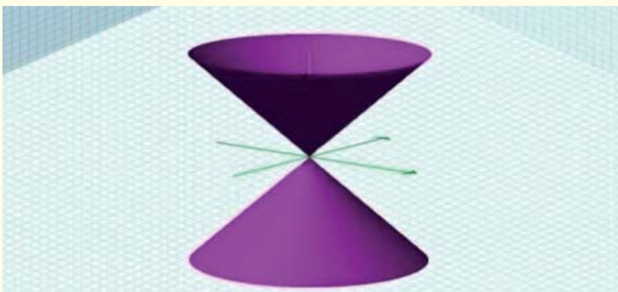


Figure 1.3. *Double cone*

Taking different planes, slice through the double cone and hence draw the shape that is obtained when the plane:

1. Is parallel to a generator but not along the generator.
2. Cuts the cone obliquely.
3. Is parallel to the axis but not along the axis.
4. Is parallel to the base but does not pass through the vertex.

Conic is the name given to the shapes that we obtain by taking different plane slices through a double cone. The sections of a right circular cone by different planes give curves of different shapes.

From activity 8.1, when different plane slices through a double cone we obtain:

a) **A parabola:** This is the section formed when the plane is parallel to a generator but not along the generator. See figure 8.4.

Figure 1.4. *Parabola*

b) **An ellipse:** This is the section formed when the plane cuts the cone obliquely; that is, cuts the axis at an angle. See figure 8.5.

Figure 1.5. *Ellipse*

c) **A hyperbola:** This is the section formed when the plane is parallel to the axis but not along the axis. Note that the hyperbola has two branches. See figure 8.6.

Figure 1.6. *Hyperbola*

d) A circle: This is a section formed when a plane is parallel to the base but does not pass through the vertex. The circle can be regarded as a special case of an ellipse. See figure 8.7.

Figure 1.7. *A circle*

Definition

A conic section is the set of all points which move in a plane such that its distance from a fixed point and a fixed straight line not containing the fixed point are in a **constant ratio**.

We use the term **degenerate conic sections** to describe the single point, single straight line and the term **non-degenerate conic sections** to describe parabola, ellipse or hyperbola.

The three non-degenerate conics (the parabola, ellipse and hyperbola) can be defined as the set of points P in the plane that satisfy the following condition:

The distance from a fixed point F (called the **focus** of the conic) to point variable P is a constant multiple of distance from a fixed straight line (called its **directrix)** to point P. This constant multiple is called its **eccentricity,** e**.**

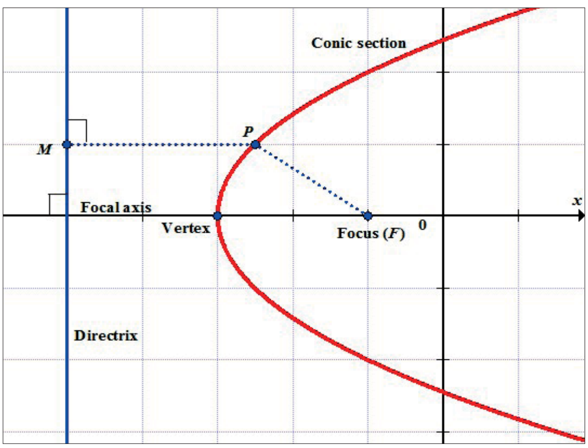


Figure 1.8. *Conic section*

From figure 8.8, we have $\left|\overline{PF}\right| = e\left|\overline{PM}\right|$ where M is a foot of perpendicularity of line joining P to directrix, point P lying on conic and F the focal point.

A focal axis is a line passing through the focus and perpendicular to the directrix.

A vertex is a point where the conic intersects its axis.

A parabola has one focus and one directrix while ellipse and hyperbola have two foci and two directrices.

Notice

The different conics arise according to the value of eccentricity e.

A non-degenerate conic is:

- **An ellipse** if $0 \le e < 1$
- **A parabola** if $e = 1$
- **A hyperbola** if $e > 1$

When $e = 0$, the ellipse is actually a circle whose focus is the centre of the **circle** and the directrix is at infinity.

Any conic section is represented by the second degree equation

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

where A, B, C, D, E and F are real numbers and A, B, C are not all nulls.

Application activity 8.1

Taking different planes, slice through a double cone and explain how to obtain:

1. A single point
2. A single line
3. Pair of lines

8.2. Parabola

8.2.1. Definition and equation

Activity 8.2

1. What is the equation of the locus of all points equidistant from the two points $(5,3)$ and $(2,1)$? Does the equation obtained represent a straight line or a curve?
2. Find the equation of the curve that is the locus of all points equidistant from the line $x = -3$ and the point $(3,0)$.
3. In each case 1) and 2), plot the curve or straight line.

A **parabola** is the set of points $P(x, y)$ in the plane equidistant from a fixed point F, called focus and a fixed line d, called the **directrix** i.e. $PF = 1 \cdot PD$, where D is a point of directrix d.

Simply, the parabola is the set $C = \left\{P(x, y) : \overline{PF} = \overline{PD}\right\}$.

We obtain the equation of a parabola in standard form if we choose:

a) The focus F on the $x-axis$ to have coordinate $(a, 0)$.

b) The directrix d to be the line with equation $x = -a$.

c) The $x-axis$ is called the axis of the parabola (axis of symmetry).

d) The origin is the vertex of the parabola.

e) Parabola has no centre.

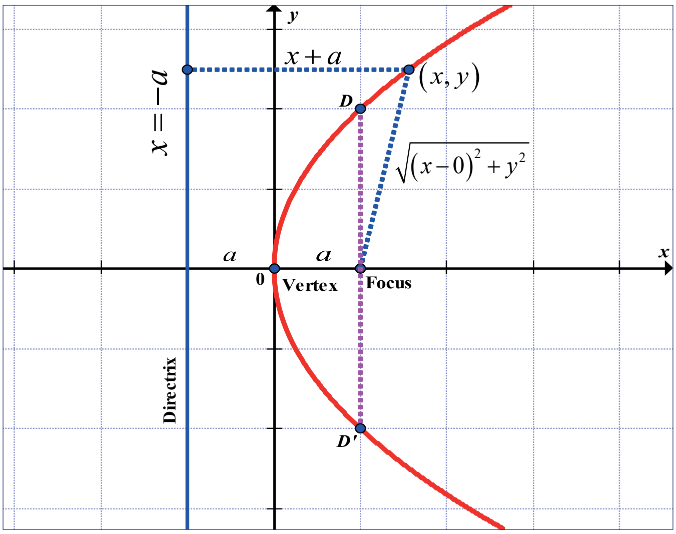


Figure 1.9. *Characteristics of parabola*

From figure 8.9, the distance from point (x, y) to the focus $(a, 0)$ is $\sqrt{(x-a)^2 + y^2}$.

The distance from point (x, y) to the line (directrix) $x = -a$ is $x + a$.

Since these two distances are equal, we have $\sqrt{(x-a)^2 + y^2} = x + a$.

Squaring both sides and expanding, we get

$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$.

Combining like terms yields

$y^2 = 4ax$

Thus, the standard equation of a parabola, whose focus at point $(a,0)$ and directrix with equation $x=-a$, is given by

$y^2 = 4ax$.

Notice

- If the parabola has vertex at (h,k), then the equation is $(y-k)^2 = 4a(x-h)$..
- Parametric equations of parabola are

$$\begin{cases} x = at^2 \\ y = 2at \end{cases} \text{ where } t \text{ is a parameter.}$$

- The equation of a parabola, whose focus is at point $(0,a)$ and directrix has the equation $y=-a$, is given by $x^2 = 4ay$.
- Recall that the distance from point (m,n) onto the line $ax+by=c$ is given by $\left|\dfrac{am+bn-c}{\sqrt{a^2+b^2}}\right|$.

Definitions

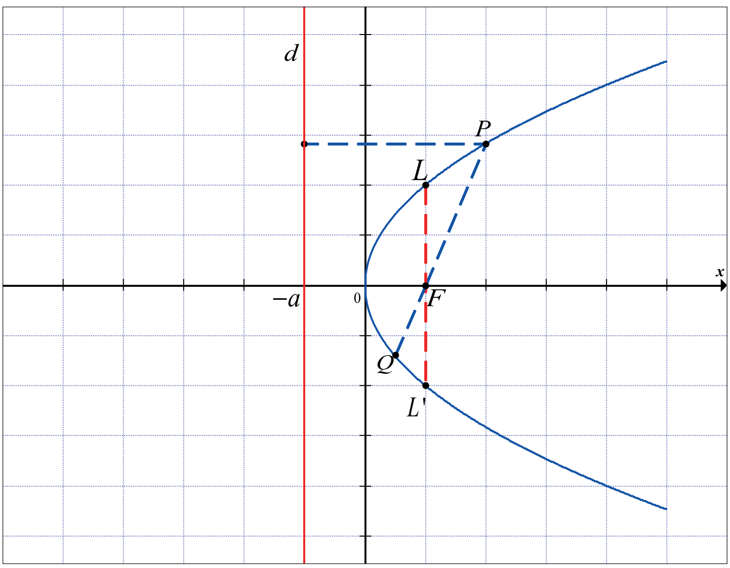


Figure 1.10. *Focal chord and latus rectum of a parabola*

- The distance from a point on a parabola to its focus is called the **focal distance** of the point. In figure 8.10, FP is the focal distance.

- A chord of the parabola which passes through its focus is called the **focal chord**. In figure 8.10, PQ is the focal chord.
- The chord through the focus and perpendicular to the axis of the parabola is called the **latus rectum** of the parabola. In figure 8.10, LFL' is the latus rectum.

Example 8.1

For each of the parabolas, find the focus and the equation of the directrix:

a) $y^2 = 8x$ b) $x^2 = -6y$

Solution

a) The given parabola is $y^2 = 8x$ which is of the form $y^2 = 4ax$.

$\Rightarrow 4a = 8;\ a = 2$.

Coordinates of the foci, $F(a,0) = (2,0)$

Directrix is $x = -a$ i.e. $x = -2$ or $x + 2 = 0$

b) The given parabola is $x^2 = -6y$ which is of the form $x^2 = 4ay$.

$\Rightarrow 4a = -6;\ a = -\dfrac{3}{2}$.

Coordinates of the foci $F(0,a) = \left(0, -\dfrac{3}{2}\right)$

Directrix is $y = -a$ i.e. $y = \dfrac{3}{2}$ or $2y - 3 = 0$.

Example 8.2

Find the foci, vertices, directrices and axis of the following parabolas:

a) $y = -4x^2 + 3x$ b) $x^2 + 2y - 3x + 5 = 0$

In each case, sketch the parabola.

Solution

a) The given parabola is $y = -4x^2 + 3x \Rightarrow 4x^2 - 3x = -y$

$$\Leftrightarrow x^2 - \frac{3}{4}x = -\frac{1}{4}y$$

Completing squares gives

$$x^2-\frac{3}{4}x+\frac{9}{64}=-\frac{1}{4}y+\frac{9}{64} \Leftrightarrow \left(x-\frac{3}{8}\right)^2=-\frac{1}{4}y+\frac{9}{64}$$

$$\Leftrightarrow \left(x-\frac{3}{8}\right)^2=-\frac{1}{4}\left(y-\frac{9}{16}\right)$$

Shifting the origin to the point $\left(\frac{3}{8},\frac{9}{16}\right)$, yields

$X^2=-\frac{1}{4}Y$ with $X=x-\frac{3}{8}$, $Y=y-\frac{9}{16}$.

The parabola of the form $X^2=4aY$ has focus at point $(0,a)$ and directrix with equation $Y=a$.

Hence, $4a=-\frac{1}{4} \Rightarrow a=-\frac{1}{16}$.

With respect to new coordinate system

Focus $F\left(0,-\frac{1}{2}\right)$, Vertex $V(0,0)$, Directrix is $Y=\frac{1}{2}$

and Axis is $X=0$.

With respect to original coordinate system

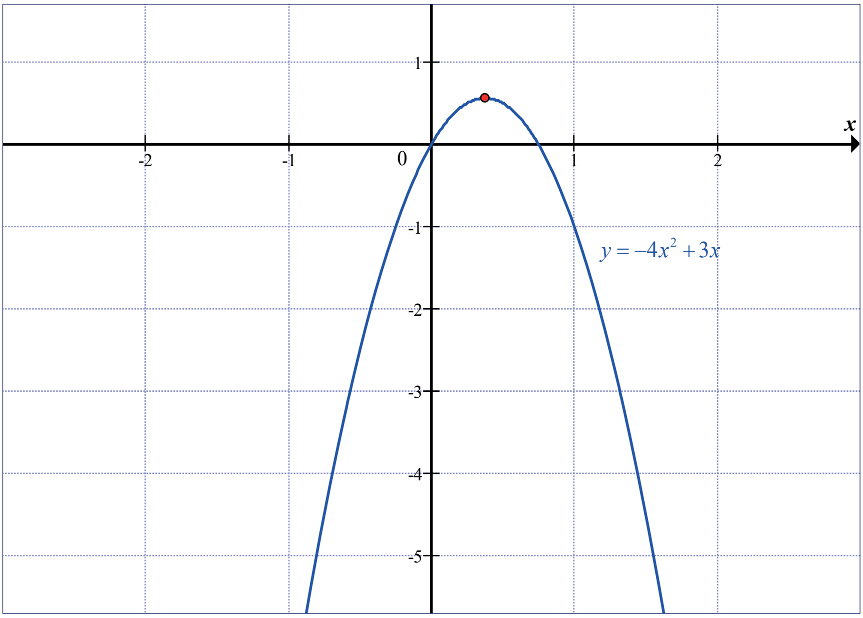
Focus $F\left(0+\frac{3}{8},-\frac{1}{16}+\frac{9}{16}\right)$ or $F\left(\frac{3}{8},\frac{1}{2}\right)$

Vertex $V\left(0+\frac{3}{8},0+\frac{9}{16}\right)$ or $V\left(\frac{3}{8},\frac{9}{16}\right)$

Directrix is $y=\frac{1}{16}+\frac{9}{16}$ or $y=\frac{5}{8}$

Axis is $x=0+\frac{3}{8}$ or $x=\frac{3}{8}$.

Sketch



b) The given parabola is $x^2+2y-3x+5=0$

$$\Leftrightarrow x^2-3x=-2y-5$$

Adding $\frac{9}{4}$ for completing square gives

$$\left(x-\frac{3}{2}\right)^2=-2y-5+\frac{9}{4}\Leftrightarrow\left(x-\frac{3}{2}\right)^2=-2y-\frac{11}{4}$$

$$\Leftrightarrow\left(x-\frac{3}{2}\right)^2=-2\left(y+\frac{11}{8}\right)$$

Shifting the origin to the point $\left(\frac{3}{2},-\frac{11}{8}\right)$, we get

$X^2=-2Y$ with $X=x-\frac{3}{2}$, $Y=y+\frac{11}{8}$.

The parabola $X^2=-2Y$ is of the form $X^2=4aY$.

Thus, $4a=-2\Rightarrow a=-\frac{1}{2}$.

With respect to new coordinate system

Focus $F\left(0,-\dfrac{1}{2}\right)$ Vertex $V(0,0)$

Directrix is $y=\dfrac{1}{2}$ Axis is $x=0$.

With respect to original coordinate system

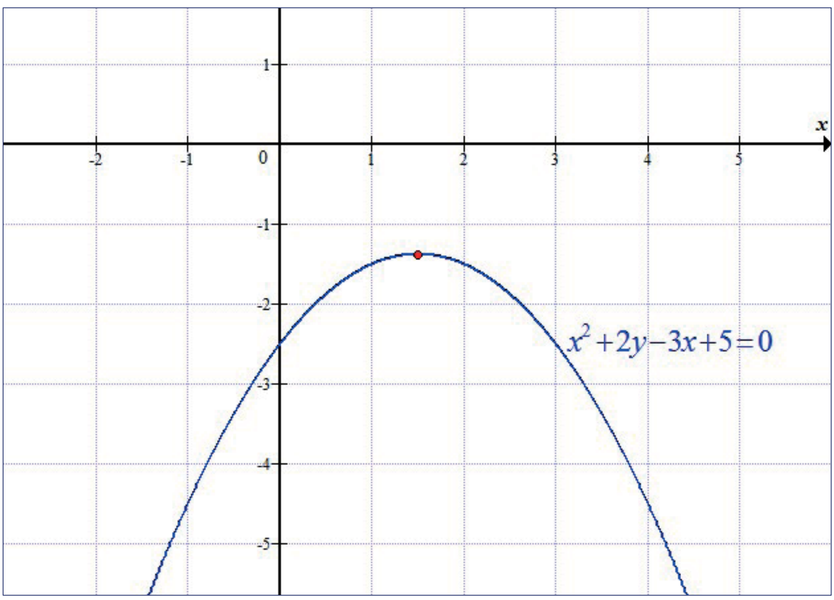
Focus $F\left(0+\dfrac{3}{2},-\dfrac{1}{2}-\dfrac{11}{8}\right)$ or $F\left(\dfrac{3}{2},-\dfrac{15}{8}\right)$

Vertex $V\left(0+\dfrac{3}{2},0+-\dfrac{11}{8}\right)$ or $V\left(\dfrac{3}{2},-\dfrac{11}{8}\right)$

Directrix is $y=\dfrac{1}{2}-\dfrac{11}{8}$ or $y=-\dfrac{7}{8}$

Axis is $x=0+\dfrac{3}{2}$ or $x=\dfrac{3}{2}$.

Sketch



Example 8.3

Find the equation of the parabola whose focus is $(5,3)$ and the directrix is the line $3x-4y+1=0$.

Solution

Let $P(x,y)$ be any point on the parabola. The focus is $F(5,3)$

Distance from point $P(x,y)$ to the directrix $3x-4y+1=0$ is

$$\frac{3x-4y+1}{\sqrt{9+16}}=\frac{3x-4y+1}{\sqrt{25}}$$

Distance from $P(x,y)$ to $F(5,3)$ is $\sqrt{(x-5)^2+(y-3)^2}$

Now, $\sqrt{(x-5)^2+(y-3)^2}=\dfrac{3x-4y+1}{\sqrt{25}}$

$$\Rightarrow\left(\sqrt{(x-5)^2+(y-3)^2}\right)^2=\left(\frac{3x-4y+1}{\sqrt{25}}\right)^2$$

$$\Rightarrow(x-5)^2+(y-3)^2=\frac{(3x-4y+1)^2}{25}$$

$$\Rightarrow 25(x^2-10x+25)+25(y^2-6y+9)=9x^2+16y^2-24xy+6x-8y+1$$

$$\Rightarrow 16x^2+9y^2+24xy-256x-142y+849=0$$

Thus, the required equation is $16x^2+9y^2+24xy-256x-142y+849=0$

Application activity 8.2

1. Find the focus and directrix of the parabola with equation $y^2=-8x$.

2. For each of the following equations, sketch the parabola. Clearly show the focus, vertex and directrix.

   a) $y^2=6x$ b) $x^2=-9y$

   c) $(y-3)^2=6(x-2)$ d) $x^2-4x+2y=1$

3. For each of the parabolas, find the focus, equation of the directrix, length of latus rectum, equation of latus rectum and ends of latus rectum:

   a) $y^2=25x$ b) $x^2=8y$ c) $x^2=-5y$

4. Determine the equation of a parabola with vertex $(1,2)$ and focus $(4,2)$.

5. Find an equation of the parabola having its:

   a) Focus $(0,-2)$, directrix $y=2$.

   b) Focus at $(-3,0)$ and the directrix $x+5=0$.

   c) Focus $(-1,-2)$ and the directrix $x-2y+3=0$.

6. For the parabola $4(y-1)^2=-7(x-3)$, find the:

   a) Latus rectum.

   b) Coordinates of the focus and the vertex.

7. Determine the point on the parabola $y^2=9x$ at which the ordinate is three times the abscissa.

8.2.2. Tangent line and normal line

Activity 8.3

1. Using the technique for the differentiation of implicit functions, derive the formula for tangent line on parabola $y^2=4ax$ at a point (x_0,y_0).
2. Deduce the equation of normal line on the parabola at the point (x,y).
3. Draw the tangent line of $y^2=2x$ to $(0,0)$.

Hint:

$$T \equiv y-y_o=m(x-x_o),\ with\ m=\left.\frac{dy}{dx}\right|_{x=x_o}$$

From activity 8.3, the tangent line at a point (x_0,y_0), on parabola $y^2=4ax$, is given by

$$T \equiv y_0y=2a(x+x_0)$$

Notice

Condition for tangency

The condition for tangency states that the line $y = mx + c$ touches the parabola $y^2 = 4ax$ if $c = \frac{a}{m}$.

In fact, let $y = mx + c$ be the line tangent to the parabola $y^2 = 4ax$, then

$$\begin{cases} y = mx + c \\ y^2 = 4ax \end{cases}$$

$$\Rightarrow (mx + c)^2 = 4ax$$

$$\Rightarrow m^2x^2 + 2mcx + c^2 = 4ax$$

$$\Rightarrow m^2x^2 + 2mcx + c^2 - 4ax = 0$$

$$\Rightarrow m^2x^2 + (2mc - 4a)x + c^2 = 0$$

The line will touch the parabola if it intersects at one point only. This will happen only when the roots are real and coincident or the discriminant of the above equation is zero.

$$m^2x^2 + (2mc - 4a)x + c^2 = 0$$

$$\Delta = (2mc - 4a)^2 - 4m^2c^2 = 0$$

$$4m^2c^2 - 16mca + 16a^2 - 4m^2c^2 = 0$$

$$\Rightarrow -16mca + 16a^2 = 0$$

$$\Rightarrow 16a^2 = 16mca$$

$$\Rightarrow c = \frac{a}{m}$$

In this case, the tangent line is $y = mx + \frac{a}{m}$.

Example 8.4

Find the tangent line to the parabola $y^2 = 8x$ at point $A(2, -4)$.

Solution

The tangent line at point (x_0, y_0), on parabola $y^2 = 4ax$, is given by

$$T \equiv y_0 y = 2a(x + x_0)$$

But, $a=2$

The tangent line is $T \equiv -4y = 4(x+2)$

$\Rightarrow T \equiv -4y = 4x+8$

$\Rightarrow T \equiv y = -x-2$

Alternative method

The tangent line at point (x_0, y_0) is $T \equiv y - y_0 = m(x - x_0)$ where m is the gradient.

Differentiating the given equation, to obtain slope of the tangent;

$$\frac{d}{dx}(y^2 = 8x) \Leftrightarrow 2y\frac{dy}{dx} = 8 \text{ or } \frac{dy}{dx} = \frac{4}{y}.$$

$$\Rightarrow \text{Slope, } m = \left.\frac{dy}{dx}\right|_{y=-4} = -1.$$

Hence, $T \equiv y+4 = -(x-2) \Leftrightarrow y = -x+2-4 \Rightarrow T \equiv y = -x-2$

Application activity 8.3

1. Determine the gradient of the curve $3xy + y^2 = -2$ at the point $(1,-2)$.
2. Find the coordinates of the focus and the vertex, the equation of the axis, directrix and the tangent at vertex for the parabola $x^2 + 4x + 4y + 16 = 0$
3. Find equation of tangents drawn from $(-2,3)$ to the curve $y^2 = 8x$.
4. Find equation of the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$. If this normal passes through the point $(6a,0)$, find the possible values of t.

8.3. Ellipse

8.3.1. Definition and equation

Activity 8.4

1. What is the equation of the curve that is the locus of all points in which the ratio of its distance from the point $(3,0)$ to its distance from the line $x=\frac{25}{3}$ is equal to $\frac{3}{5}$.
2. Sketch the curve in 1.

We define an **ellipse** with eccentricity e(where $0<e<1$) to be the set of points P in the plane whose distances from a fixed point F is e times their distances from a fixed line d.

Let us consider figure 8.11:

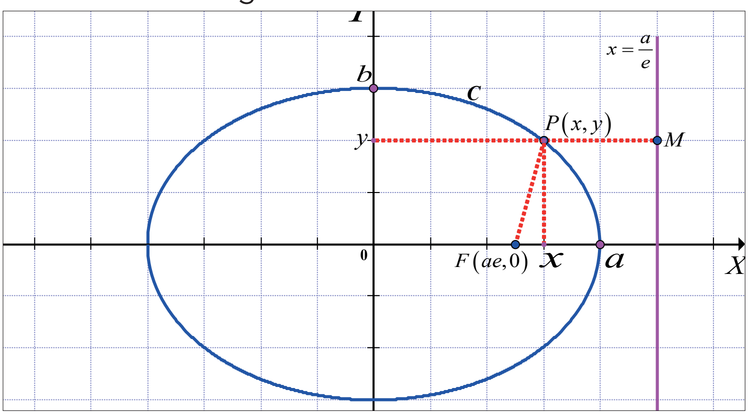


Figure 1.11. *Ellipse*

We can obtain equation of an ellipse, in standard form if we choose

1. The focus F to lie on the $x-axis$ and have coordinates $(ae,0)$.
2. The directrix d to be the line with equation $x=\frac{a}{e}$.

From activity 8.4 and figure 8.11, we get the equation of ellipse in standard form, that is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Since the equation of ellipse in standard form is symmetric about both the $x-axis$ and $y-axis$, then, there is a second focus $F'(-ae,0)$ and a second directrix $d' \equiv x = -\frac{a}{e}$.

The distance between two foci is $2ae$ with $e = \sqrt{\frac{a^2 - b^2}{a^2}}$.

Considering figure 8.12, we identify the elements of an ellipse.

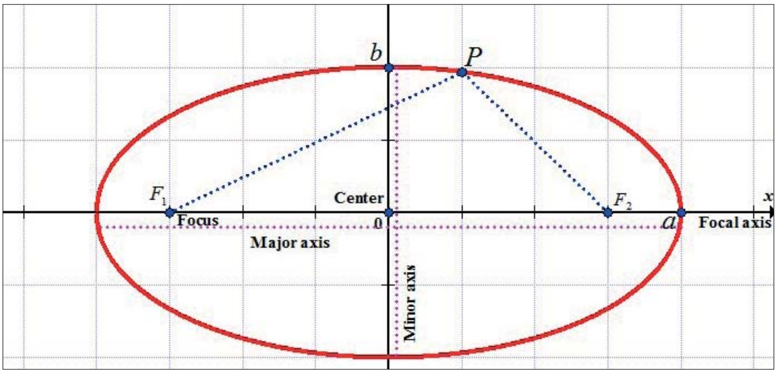


Figure 1.12. *Characteristics of ellipse*

Ellipse is also defined as the locus of points P such that the sum of the distances from P to two fixed points is constant.

Thus, let F_1 and F_2 be the two points (called **foci**, the plural of focus), then the defining relation for the ellipse is $PF_1 + PF_2 = 2a$.

Therefore, Ellipse C is the set $C = \{P(x, y) : PF_1 + PF_2 = 2a, a \in \mathbb{R}\}$.

The line through the foci is called the **focal axis** of the ellipse; the point on the focal axis halfway between the foci is called the **centre of the ellipse**; the points where the ellipse crosses the focal axis are called the **vertices**.

The line segment joining the two vertices is called the **major axis**, and the line segment through the centre and perpendicular to the major axis, with both end-points on the ellipse, is called the **minor axis**.

If the ellipse has centre at (h, k) which is not the origin, then the equation is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

For an ellipse centre of origin h=k=0.

Parametric equations of ellipse whose centre (h, k) are

$\begin{cases} x = h + a\cos t \\ y = k + b\sin t \end{cases}$ where t is a parameter and $t \in [-\pi, \pi]$.

Length of latus rectum of ellipse

Let us consider ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a>b>0$

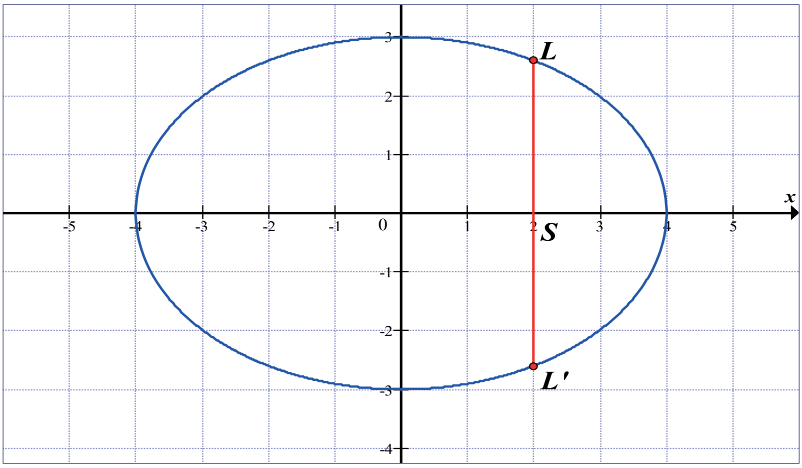


Figure 1.13. *Latus rectum of ellipse*

Let LSL' be the latus rectum through S in figure 8.13.

Let $SL=l$; thus $L(ae,l)$.

Since the point (ae,l) lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

Therefore,

$$\frac{(ae)^2}{a^2}+\frac{l^2}{b^2}=1 \Leftrightarrow \frac{a^2e^2}{a^2}+\frac{l^2}{b^2}=1 \Leftrightarrow e^2+\frac{l^2}{b^2}=1$$

$$\Leftrightarrow \frac{l^2}{b^2}=1-e^2 \Leftrightarrow \frac{l^2}{b^2}=\frac{b^2}{a^2} \text{ as } \left[b^2=a^2\left(1-e^2\right)\right]$$

$$\Leftrightarrow l^2=\frac{b^4}{a^2} \Rightarrow l=\frac{b^2}{a}$$

Hence, the **length of latus rectum** is $2l=\frac{2b^2}{a}$.

Ends of latus rectum through S are $\left(ae,\frac{b^2}{a}\right)$ and $\left(ae,-\frac{b^2}{a}\right)$.

Also, the **ends of latus rectum** through S' are $\left(-ae, \frac{b^2}{a}\right)$ and $\left(-ae, -\frac{b^2}{a}\right)$.

Equations of latus rectum through S and S' are $x = ae$ and $x = -ae$ respectively.

Notice

If the denominator of y^2 is greater than the denominator of x^2, the major axis is vertical and the minor axis is horizontal. Always, we will take $b \le a$. Here, the equation is written as $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$. In this case, foci are $F'(0, \pm ae)$,

directrices are $d' \equiv y = \pm \frac{e}{a}$ and vertices are $(0, \pm a)$.

Example 8.5

Find the equation for the ellipse whose one focus is $(2,1)$ and its corresponding directrix is the line $2x - y + 3 = 0$ and the eccentricity is $\frac{\sqrt{2}}{2}$.

Solution

Let $P(x, y)$ represent any point on this ellipse and e be the eccentricity

$$e = \frac{\text{distance from point } P \text{ to focus}}{\text{distance from point } P \text{ to directrix}}$$

$$\therefore \frac{\sqrt{2}}{2} = \frac{\sqrt{(x-2)^2 + (y-1)^2}}{\frac{2x - y + 3}{\sqrt{4+1}}} \Rightarrow \frac{\sqrt{2}}{2} = \sqrt{(x-2)^2 + (y-1)^2} \times \frac{\sqrt{5}}{2x - y + 3}$$

$$\Rightarrow \left(\frac{\sqrt{2}}{2}\right)^2 = \left(\frac{\sqrt{5}\sqrt{(x-2)^2 + (y-1)^2}}{2x - y + 3}\right)^2$$

$$\Rightarrow \frac{2}{4} = \frac{5\left[(x-2)^2 + (y-1)^2\right]}{(2x - y + 3)^2} \Rightarrow 2(2x - y + 3)^2 = 20\left[(x-2)^2 + (y-1)^2\right]$$

Expanding and simplifying we get

$6x^2 + 4xy + 9y^2 - 52x - 14y + 41 = 0$

Example 8.6

Find the centre, the length of axes and eccentricity of the ellipse $2x^2 + 3y^2 - 4x - 12y + 13 = 0$.

Solution

By completing squares we have

$$2x^2 + 3y^2 - 4x - 12y + 13 = 0$$

$$\Rightarrow 2\left(x^2 - 2x\right) + 3\left(y^2 - 4y\right) + 13 = 0$$

$$\Rightarrow 2\left(x^2 - 2x\right) + 3\left(y^2 - 4y\right) + 13 = 0$$

$$\Rightarrow 2\left[\left(x-1\right)^2 - 1\right] + 3\left[\left(y-2\right)^2 - 4\right] + 13 = 0$$

$$\Rightarrow 2\left(x-1\right)^2 - 2 + 3\left(y-2\right)^2 - 12 + 13 = 0 \quad \Rightarrow 2\left(x-1\right)^2 + 3\left(y-2\right)^2 = 1$$

$$\Rightarrow \frac{\left(x-1\right)^2}{\frac{1}{2}} + \frac{\left(y-2\right)^2}{\frac{1}{3}} = 1$$

The centre is $\left(1,2\right)$

$$a^2 = \frac{1}{2} \Rightarrow a = \frac{\sqrt{2}}{2} \text{ and } b^2 = \frac{1}{3} \Rightarrow b = \frac{\sqrt{3}}{3}$$

Major axis is $2a = \sqrt{2}$, minor axis is $2b = \frac{2\sqrt{3}}{3}$

Also, we know that $b^2 = a^2\left(1 - e^2\right)$ where e is the eccentricity

$$\frac{1}{3} = \frac{1}{2}\left(1 - e^2\right) \Rightarrow e^2 = 1 - \frac{2}{3} \Rightarrow e^2 = \frac{1}{3} \Rightarrow e = \frac{\sqrt{3}}{3}$$

Example 8.7

Find the length of the axes, eccentricity, coordinates of foci, equation of directrices and latus rectum for each of the following ellipses:

a) $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$ b) $9x^2+5y^2=45$

Solution

a) For ellipse $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$, $a^2=25, b^2=16$ and $a>b$.

Therefore,

Length of the axes are $2a=10$ and $2b=8$.

Eccentricity $e=\sqrt{\dfrac{a^2-b^2}{a^2}}=\sqrt{\dfrac{25-16}{25}}=\sqrt{\dfrac{9}{25}}=\dfrac{3}{5}$

Coordinates of foci are $(ae,0)$ and $(-ae,0)$

But $ae=5\left(\dfrac{3}{5}\right)=3$.

Hence, coordinates of foci are $(3,0)$ and $(-3,0)$.

Equation of directrices are $x=\dfrac{a}{e}$ and $x=-\dfrac{a}{e}$

Hence, equation of directrices are $x=\pm\dfrac{5}{\frac{3}{5}}$ or $x=\pm\dfrac{25}{3}$.

Latus rectum: $l=\dfrac{2b^2}{a}=2\times\dfrac{16}{5}=\dfrac{32}{5}$

b) Ellipse $9x^2+5y^2=45$ can be rewritten as $\dfrac{x^2}{5}+\dfrac{y^2}{9}=1$

$a^2=5, b^2=9$ and $a<b$.

Major axis is $2b=6$

Minor axis is $2a=2\sqrt{5}$.

Eccentricity $e=\sqrt{\dfrac{b^2-a^2}{b^2}}=\sqrt{\dfrac{9-5}{9}}=\sqrt{\dfrac{4}{9}}=\dfrac{2}{3}$

Coordinates of foci are $(0,be)$ and $(0,-be)$

But $be=3\left(\dfrac{2}{3}\right)=2$.

Hence, coordinates of foci are $(0,2)$ and $(0,-2)$.

Equation of directrices are $y=\frac{b}{e}$ and $y=-\frac{b}{e}$.

Thus, equation of directrices are $y=\pm\frac{3}{\frac{2}{3}}$ or $y=\pm\frac{9}{2}$.

Latus rectum is $\frac{2a^2}{b}=2\times\frac{5}{3}=\frac{10}{3}$

Application activity 8.4

1. Find the foci of the ellipse $2x^2+y^2=4$.
2. Find the equation of the ellipse passing through $(1,4)$ and $(-6,-1)$.
3. Find an equation for the ellipse with foci $(0,\pm 2)$ and major axis with end-points $(0,\pm 4)$.
4. For each of the following equations, sketch the ellipse. Label the foci, the ends of the major and minor axes.

   a) $\frac{x^2}{16}+\frac{y^2}{9}=1$ b) $3(x+2)^2+4(y+1)^2=12$

   c) $4x^2+y^2+8x-10y+13=0$
5. By completing squares, show that the curve $16x^2+9y^2-64x-54y+1=0$ is an ellipse and hence deduce the foci.
6. Find the eccentricity of the ellipses whose:

   a) $\text{Latus rectum}=\frac{1}{2}\text{major axis}$.

   b) $\text{Distance between directrices}=3\text{distance between foci}$.

   c) $\text{Latus rectum}=\text{semi}-\text{minor axis}$.
7. Find the focal distance of the point $P(5,4\sqrt{3})$ on the ellipse $16x^2+25y^2=1600$.
8. The ellipse $\frac{x^2}{169}+\frac{y^2}{25}=1$ has the same eccentricity as the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Find the ratio of a to b.

8.3.2. Tangent line and normal line

Activity 8.5

1. Using the derivative of implicit functions, derive the equation of tangent line to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at point (x_0, y_0).

   Hint:

   $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Leftrightarrow b^2x^2+a^2y^2=a^2b^2$$

2. Draw the tangent line to the curve $x^2+\frac{y^2}{9}=1$ at $(0,3)$.

From activity 8.5, the tangent line at point (x_0, y_0), on ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is given by: $T \equiv \frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1$.

Recall that, if m is the gradient of tangent line T, the gradient of the normal line N at the same point is $-\frac{1}{m}$.

Notice

Condition of tangency

The condition of tangency states that the line $y=mx+c$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ if $c^2=b^2+a^2m^2$.

In fact, let $y=mx+c$ be the line tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then

$$\begin{cases} y=mx+c \\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \end{cases}$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=a^2b^2$$

$$\Rightarrow b^2x^2+a^2\left(m^2x^2+2mcx+c^2\right)=a^2b^2$$

$$\Rightarrow b^2x^2+a^2m^2x^2+2a^2mcx+a^2c^2-a^2b^2=0$$

$$\Rightarrow\left(b^2+a^2m^2\right)x^2+2a^2mcx+a^2c^2-a^2b^2=0$$

The line will touch the ellipse if it intersects at one point only. This will happen only when the roots are real and coincident or the discriminant of the above equation is zero.

$$\left(b^2+a^2m^2\right)x^2+2a^2mcx+a^2c^2-a^2b^2=0$$

$$\Delta=\left(2a^2mc\right)^2-4\left(b^2+a^2m^2\right)\left(a^2c^2-a^2b^2\right)=0$$

$$4a^4m^2c^2-4b^2a^2c^2+4a^2b^4-4a^4m^2c^2+4a^4m^2b^2=0$$

$$\Rightarrow -4b^2a^2c^2+4a^2b^4+4a^4m^2b^2=0$$

$$\Rightarrow 4b^2a^2\left(-c^2+b^2+a^2m^2\right)=0$$

$$\Rightarrow -c^2+b^2+a^2m^2=0$$

$$\Rightarrow c^2=b^2+a^2m^2$$

$$4a^4m^2c^2-4b^2a^2c^2+4a^2b^4-4a^4m^2c^2+4a^4m^2b^2=0$$

$$\Rightarrow -4b^2a^2c^2+4a^2b^4+4a^4m^2b^2=0$$

$$\Rightarrow 4b^2a^2\left(-c^2+b^2+a^2m^2\right)=0$$

$$\Rightarrow -c^2+b^2+a^2m^2=0$$

$$\Rightarrow c^2=b^2+a^2m^2$$

Thus $c=\pm\sqrt{b^2+a^2m^2}$

In this case, the tangent line is $y=mx\pm\sqrt{b^2+a^2m^2}$

Example 8.8

Determine the equation of tangent and normal line to ellipse $\dfrac{x^2}{4}+\dfrac{y^2}{9}=1$ at point $\left(\dfrac{6}{5},\dfrac{12}{5}\right)$

Solution

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

The point $\left(\frac{6}{5}, \frac{12}{5}\right)$ lies on $\frac{x^2}{4}+\frac{y^2}{9}=1$ since

$$\frac{\frac{36}{25}}{4}+\frac{\frac{144}{25}}{9}=\frac{36}{100}+\frac{144}{225}=\frac{9}{25}+\frac{16}{25}=1$$

Equation of tangent line:

Since $T \equiv y-y_o=m(x-x_o)$ where $m=\left.\frac{dy}{dx}\right|_{x=x_o}$

differentiating the equation of the ellipse with respect to x gives

$$\frac{2x}{4}+\frac{2}{9}y\frac{dy}{dx}=0 \Leftrightarrow \frac{x}{2}+\frac{2}{9}y\frac{dy}{dx} \text{ or } \frac{dy}{dx}=-\frac{9}{4}\frac{x}{y}$$

At point $\left(\frac{6}{5}, \frac{12}{5}\right)$, $m=-\frac{dy}{dx}=-\frac{9}{4}\times\frac{\frac{6}{5}}{\frac{12}{5}}=-\frac{9}{4}\times\frac{6}{12}=-\frac{9}{8}$

Equation of tangent line is

$$T \equiv y-\frac{12}{5}=-\frac{9}{8}\left(x-\frac{6}{5}\right) \Leftrightarrow 40y-96=-45\left(x-\frac{6}{5}\right)$$

$$\Leftrightarrow 40y-96=-45x+54 \Leftrightarrow 45x+40y=150 \Leftrightarrow 9x+8y-30=0$$

Therefore, $T \equiv 9x+8y-30=0$.

Alternative method

Since $T \equiv \frac{x_o x}{a^2}+\frac{y_o y}{b^2}=1$, $x_o=\frac{6}{5}$ and $y_o=\frac{12}{5}$, then

$$T \equiv \frac{\frac{6}{5}x}{4}+\frac{\frac{12}{5}y}{9}=1 \Leftrightarrow \frac{6x}{20}+\frac{12y}{45}=1 \Leftrightarrow \frac{3x}{10}+\frac{4y}{15}=1 \Leftrightarrow 45x+40y=150$$

$$\Leftrightarrow 9x+8y-30=0$$

Hence, $T \equiv 9x+8y-30=0$.

Normal line equation:

$$N \equiv y - y_o = -\frac{1}{m}(x - x_o) \text{ where } m = \left.\frac{dy}{dx}\right|_{x=x_o}$$

the slope of normal line to the $\frac{x^2}{4}+\frac{y^2}{9}=1$ at point $\left(\frac{6}{5},\frac{12}{5}\right)$ is $-\frac{1}{-\frac{9}{8}}=\frac{8}{9}$.

$\Rightarrow$ The equation of normal line is $N \equiv y-\frac{12}{5}=\frac{8}{9}\left(x-\frac{6}{5}\right)$

$\Leftrightarrow 45y-108=40\left(x-\frac{6}{5}\right) \Leftrightarrow 45y-108=40x-48$

$\Leftrightarrow -40x+45y-60=0 \Leftrightarrow 8x-9y+12=0$

Therefore, the equation of this normal line is $N \equiv 8x-9y+12=0$

Application activity 8.5

1. Find the gradients of the tangents drawn to the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=2$ at the point where $x=2$.
2. Determine the equation of the tangent drawn to the ellipse $x=3\cos\theta, y=2\sin\theta$ at a point where $\theta=\frac{\pi}{6}$.
3. Find the value(s) of k such that the line $x+2y=k$ is tangent to the ellipse $x^2+4y^2=8$. For each value of k, determine the point of contact.
4. Find equations for the tangents to the ellipse $(x-2)^2+(y-1)^2=5$ at the points where the ellipse cuts the coordinates axes.
5. Find the range of values of m so that the line $y=mx$ and the conic $x^2+y^2-6x-8y+24=0$
   a) Intersect at two points.
   b) Touch at two points.
   c) Do not touch each other.

8.4. Hyperbola

8.4.1. Definition and equation

Activity 8.6

If the foci of a conic are $F_1(c,0)$ and $F_2(-c,0)$ where $c=\sqrt{a^2+b^2}$, derive the equation of locus for which the difference of the distances from any point $P(x,y)$ on conic to these two foci is $2a$.

Indicate the nature of this curve by use of a sketch if necessary.

From activity 8.6, the locus of points P such that the difference of the distances from P to two fixed points (foci) is a constant i.e.
$C=\{P(x,y):PF_1-PF_2=2a, a\in\mathbb{R}\}$ and has equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and is called a **hyperbola**.

Let us consider figure 8.14 and define elements of hyperbola:

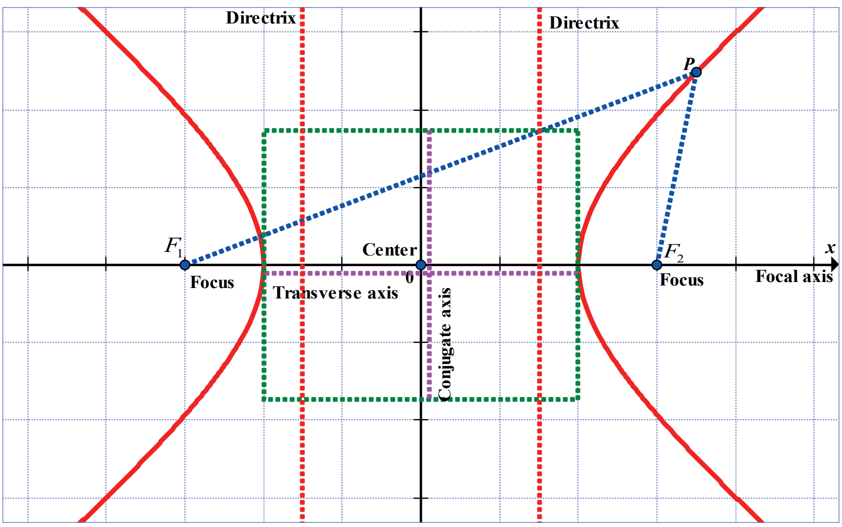


Figure 1.14. *Characteristics of hyperbola*

The line through the foci is called the **focal axis** of the hyperbola; the point on the focal axis halfway between the foci is called the **centre**; the points where the hyperbola crosses the focal axis are called the **vertices**.

The line segment joining the two vertices is called the ***transverse axis***.

This hyperbola has two asymptotes $y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$

If the hyperbola has centre at (h,k), then the equation is $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

Parametric equations of hyperbola are

$\begin{cases} x = a\sec t \\ y = b\tan t \end{cases}$ where t is a parameter and $t \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\cup \left]\frac{\pi}{2},\frac{3\pi}{2}\right[$

Since the equation of a hyperbola in standard form is symmetric about $x-axis$ and $y-axis$, there is a second focus $F'(-ae,0)$ and a second directrix $d' \equiv x = -\frac{e}{a}$.

The line segment joining the points $(\pm a,0)$, has length which is equal to $2a$ and is called a **transverse axis** and the segment joining $(0,\pm b)$, which is equal to $2b$ and is called a **conjugate axis**.

The distance between two foci is $2ae$.

Notice

Unlike in ellipse, the orientation of hyperbola is not determined by examining the relative sizes of a^2 and b^2,
but rather by noting where the minus sign occurs in the equation. If the minus sign precedes the y^2 term i.e. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$,

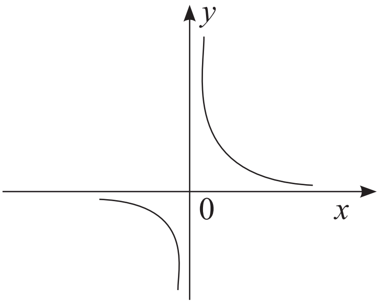
the foci lie on the $x-axis$ and if the minus sign precedes the x^2 term, i.e $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$, the foci lie on the $y-axis$, that is, the foci are $F'(0,\pm ae)$, directrices are $d' \equiv y = \pm\frac{e}{a}$ and vertices are $(0,\pm a)$.

The hyperbola $\frac{y^2}{a^2}-\frac{x^2}{b^2}=1$ has two asymptotes, $y=\frac{a}{b}x$ and $y=-\frac{a}{b}x$.

Particular case

When the eccentricity of hyperbola takes the value $\sqrt{2}$, we get the case of **perpendicular hyperbola** or **rectangular hyperbola**.

The curve of a rectangular hyperbola is shown below;



In this case, the equation is $xy = c^2$ and the parametric equations are

$$\begin{cases} x = ct \\ y = \dfrac{c}{t} \end{cases}$$ where t is a parameter different from zero.

In this case, the equation is $xy = c^2$ and the parametric equations are

$$\begin{cases} x = ct \\ y = \dfrac{c}{t} \end{cases}$$ where t is a parameter different from zero.

The asymptotes of this hyperbola are $y = \pm x$.

Example 8.9

Find, in standard form, the equation of a hyperbola whose eccentricity is $\sqrt{2}$ and the distance between foci is 16 units.

Solution

$e = \sqrt{2}$

Since we know that the distance between the two foci is $2ae$, then $2ae = 16$.

Now, $a\sqrt{2} = 8 \Rightarrow a = 4\sqrt{2} \Rightarrow a^2 = 32$

Also, $b^2 = a^2\left(e^2 - 1\right)$ or $b^2 = 32(2-1) = 32$

The equation is $\dfrac{x^2}{32} - \dfrac{y^2}{32} = 1$

Example 8.10

Find the equation of hyperbola whose one directrix is $2x+y-1=0$ and corresponding focus is $(1,2)$ and eccentricity $\sqrt{3}$.

Solution

Let $P(x,y)$ represent any point on this hyperbola.

$$e=\frac{\text{distance from point } P \text{ to focus}}{\text{distance from point } P \text{ to directrix}}$$

$$\therefore\ \sqrt{3}=\frac{\sqrt{(x-1)^2+(y-2)^2}}{\dfrac{2x+y-1}{\sqrt{4+1}}} \Rightarrow \sqrt{3}=\sqrt{(x-1)^2+(y-2)^2}\times\frac{\sqrt{5}}{2x+y-1}$$

$$\Rightarrow\left(\sqrt{3}\right)^2=\left(\frac{\sqrt{5}\sqrt{(x-1)^2+(y-2)^2}}{2x+y-1}\right)^2 \Rightarrow 3=\frac{5\left[(x-1)^2+(y-2)^2\right]}{(2x+y-1)^2}$$

$$\Rightarrow 3(2x-y+3)^2=5\left[(x-1)^2+(y-2)^2\right]$$

Expanding and simplifying we get

$$7x^2+12xy-2y^2-2x+14y-22=0$$

Example 8.11

Find the eccentricity and coordinates of foci for hyperbola $4x^2-9y^2=36$

Solution

$$4x^2-9y^2=36\Rightarrow\frac{x^2}{9}-\frac{y^2}{4}=1$$

$$a^2=9\Rightarrow a=3$$

$$b^2=4\Rightarrow b=2$$

$$b^2=a^2\left(e^2-1\right)\Rightarrow 4=9\left(e^2-1\right)\Rightarrow e^2=1+\frac{4}{9}=\frac{13}{9}\Rightarrow e=\frac{\sqrt{13}}{3}$$

Thus, the eccentricity is $\frac{\sqrt{13}}{3}$

Coordinates of foci; $(\pm ae,0)=(\pm\sqrt{13},0)$

Application activity 8.6

1. For each of the following equations, sketch the hyperbola, state the coordinates of vertices and foci, and find equations for the asymptotes.

   a) $\frac{y^2}{4}-\frac{x^2}{9}=1$ b) $\frac{(x-2)^2}{9}-\frac{(y-4)^2}{4}=1$

   c) $\frac{(y+3)^2}{36}-\frac{(x+2)^2}{4}=1$

2. Find the foci, the vertices, and asymptotes of the hyperbola $\frac{y^2}{16}-\frac{x^2}{9}=1$.

3. For the following hyperbolas, find the lengths of transverse and conjugate axes, eccentricity, coordinates of foci and vertices.

   a) $16x^2-9y^2-144=0$ b) $2x^2-3y^2=6$

   c) $y^2-16x^2=16$

4. Find the equation of the hyperbola with vertices $(0,\pm 8)$ and asymptotes $y=\pm\frac{4}{3}x$.

5. By completing squares, show that the curve $x^2-y^2-4x+8y-21=0$ is a hyperbola and hence determine the coordinates of foci, vertices and asymptote equations.

6. Find the equation of hyperbola whose;

   a) eccentricity is $\frac{5}{4}$, one of the foci is at $(2,0)$ and the corresponding directrix is $4x-3y=1$.

   b) focus is $(-3,3)$, the corresponding directrix is the line $5x+6=0$ and eccentricity is $\frac{5}{4}$.

7. Find the equation of hyperbola whose distance between two foci is 10 and eccentricity is $\frac{5}{2}$.

8.4.2. Tangent line and normal line

Example 8.12

Derive the general equation of tangent line to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at point (x_0, y_0).

From activity 8.7,

The tangent line at point (x_0, y_0) on hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is given by

$$T \equiv \frac{x_0 x}{a^2}-\frac{y_0 y}{b^2}=1.$$

Remember that if m is the gradient of tangent line T, the gradient of the normal line N at the same point is $-\frac{1}{m}$.

Notice

Condition of tangency

The condition of tangency states that the line $y = mx + c$ touches the ellipse $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ if $c^2 = a^2m^2 - b^2$.

In fact, let $y = mx + c$ be the line tangent to the ellipse

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, then

$$\begin{cases} y = mx + c \\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \end{cases}$$

$$\Rightarrow \frac{x^2}{a^2}-\frac{(mx+c)^2}{b^2}=a^2b^2 \Rightarrow b^2x^2 - a^2\left(m^2x^2+2mcx+c^2\right)=a^2b^2$$

$$\Rightarrow b^2x^2 - a^2m^2x^2 - 2a^2mcx - a^2c^2 - a^2b^2 = 0$$

$$\Rightarrow \left(b^2 - a^2m^2\right)x^2 - 2a^2mcx - a^2c^2 - a^2b^2 = 0$$

The line will touch the hyperbola if it intersects at one point only. This will happen only when the roots are real and coincident or the discriminant of the above equation is zero.

$$\left(b^2 - a^2m^2\right)x^2 - 2a^2mcx - a^2c^2 - a^2b^2 = 0$$

$$\Delta = \left(2a^2mc\right)^2 - 4\left(b^2 - a^2m^2\right)\left(-a^2c^2 - a^2b^2\right) = 0$$

$$4a^4m^2c^2 + 4b^2a^2c^2 + 4a^2b^4 - 4a^4m^2c^2 - 4a^4m^2b^2 = 0$$

$$\Rightarrow 4b^2a^2c^2 + 4a^2b^4 - 4a^4m^2b^2 = 0$$

$$\Rightarrow 4b^2a^2\left(c^2 + b^2 - a^2m^2\right) = 0 \Rightarrow c^2 + b^2 - a^2m^2 = 0$$

$$\Rightarrow c^2 = -b^2 + a^2m^2 \Rightarrow c^2 = a^2m^2 - b^2$$

Thus, $c = \pm\sqrt{a^2m^2 - b^2}$

In this case, the tangent line is $y = mx \pm \sqrt{a^2m^2 - b^2}$.

Example 8.13

Determine the value of a^2 such that the line $5x - 4y - 16 = 0$ is a tangent to the hyperbola $9x^2 - a^2y^2 = 9a^2$.

Solution

Rewriting the equation $9x^2 - a^2y^2 = 9a^2$ as $\frac{x^2}{a^2} - \frac{y^2}{9} = 1$

and the equation $5x - 4y - 16 = 0$ as $-4y = -5x + 16$ or $y = \frac{5}{4}x - 4$

But the tangency condition is $c^2 = a^2m^2 - b^2$ with $c = -4,\ m = \frac{5}{4},\ b^2 = 9$

$$\Rightarrow 16 = a^2\left(\frac{5}{4}\right)^2 - 9 \Rightarrow a^2\left(\frac{25}{16}\right) = 25 \Rightarrow a^2 = \frac{25\times16}{25} = 16$$

Thus, $a^2 = 16$

Example 8.14

Find the equations of normal lines to the hyperbola $3x^2 - 4y^2 = 12$ which are parallel to the line $-x+y=0$.

Solution

Rewriting the equation $3x^2-4y^2=12$ as $\frac{x^2}{4}-\frac{y^2}{3}=1$

and the equation $-x+y=0$ as $y=x$.

Since the normal is parallel to the line $y=x$, then the tangent is perpendicular to the line $y=x \Rightarrow$ tangent has equation of

the form $y=-x+c$ and normal line has the form $y=x+k$.

Since the tangency condition is $c^2=a^2m^2-b^2$ with
$c=0,\ m=1,\ a^2=4,\ b^2=3$

then,

$c^2=4\times1-3 \Rightarrow c=\pm1$

Then, the tangent lines are $y=-x+1$ and $y=-x+1$

We use these two lines to find the points of tangency:

For the line $y=-x+1$:

$$\begin{cases}3x^2-4y^2=12\\ y=-x+1\end{cases}$$

$$\Rightarrow 3x^2-4(-x+1)^2=12 \Rightarrow 3x^2-4\left(x^2-2x+1\right)=12$$

$$\Rightarrow 3x^2-4x^2+8x-4-12=0 \Rightarrow -x^2+8x-16=0$$

$$\Rightarrow x^2-8x+16=0 \Rightarrow x=4$$

$$y=-x+1=-3$$

Thus, point of contact between the tangent and the hyperbola is (4,-3).

For the normal line $y=x+k \Leftrightarrow -3=4+k \Rightarrow k=-7$

The normal line is $y=x-7$

Again, for the line $y=-x-1$:

$$\begin{cases}3x^2-4y^2=12\\ y=-x-1\end{cases}$$

$$\Rightarrow 3x^2-4(-x-1)^2=12 \Rightarrow 3x^2-4\left(x^2+2x+1\right)=12$$

$$\Rightarrow 3x^2-4x^2-8x-4-12=0 \Rightarrow -x^2-8x-16=0$$

$$\Rightarrow x^2+8x+16=0 \Rightarrow x=-4$$

$$y=-x-1=3$$

For the normal line $y=x+k \Leftrightarrow 3=-4+k \Rightarrow k=7$

The normal line is $y=x+7$

Example 8.15

Find the coordinates of the point at which the normal line to the curve $xy=8$ at the point $(4,2)$ cuts the tangent to the curve $16x^2-y^2=64$ at point $\left(2\frac{1}{2},6\right)$.

Solution

Normal line $N \equiv y-y_o=-\frac{1}{m}(x-x_o)$ where $m=\left.\frac{dy}{dx}\right|_{x=x_o}$

Differentiating $xy=8$ on both sides with respect to x yields

$$y+x\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{y}{x}.$$

At the point $(4,2)$, the gradient of the tangent is $m=-\frac{2}{4}=-\frac{1}{2}$.

Therefore, the gradient of the normal at point $(4,2)$ is $-\frac{1}{m}=2$.

Hence, equation of normal is $N \equiv y-2=2(x-4)$ or $y=2x-6$.

Differentiating $16x^2-y^2=64$ on both sides with respect to x yields

$$32x-2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{16x}{y}.$$

At point $\left(2\frac{1}{2},6\right)$, the gradient of the tangent is

$$m=\frac{16}{6}\left(2\frac{1}{2}\right)=\frac{20}{3}.$$

Therefore, the equation of tangent line is $T \equiv y-6=\frac{20}{3}\left(x-2\frac{1}{2}\right)$

$\Leftrightarrow 3y-18=20x-50 \Leftrightarrow 20x-3y-32=0$

To get intersection point, we solve the simultaneous equations

$$\begin{cases} y=2x-6 \\ 20x-3y-32=0 \end{cases}$$

which gives $x=1$ and $y=-4$.

Thus the required normal and tangent intersect at the point $(1,-4)$.

Application activity 8.7

1. Evaluate $\frac{dy}{dx}$ at $\theta=\frac{\pi}{6}$ radians for the hyperbola whose parametric equations are $x=3\sec\theta, y=6\tan\theta$.
2. Determine the equation of the tangent drawn to the rectangular hyperbola $x=5t, y=\frac{5}{t}$ at $t=2$.
3. Find the equation of the tangent to the curve $9x^2-y=9$ at the point $\left(-\frac{5}{3},4\right)$.
4. A line tangent to the hyperbola $4x^2-y^2=36$ intersects the $y-axis$ at the point $(0,4)$. Find the point(s) of contact between the tangent and the hyperbola.

8.5. Polar coordinates

8.6.1. Definition

Activity 7.1

1. Find the modulus of each of the following complex numbers:

   a) $z=3+4i$ b) $z=1-i$

2. Consider the complex number $z=1+i$. Find the value of θ such that $\cos\theta=\frac{1}{\sqrt{2}}$ and $\sin\theta=\frac{1}{\sqrt{2}}$ for $-\pi<\theta\le\pi$

To form a polar coordinate system in the plane, we fix a point 0 called the **pole** (or origin) and construct from 0 an initial ray called the **polar axis**. Then each point P in the plane can be assigned **polar coordinates** (r,θ) as follows:

r is the directed distance from 0 to P.

θ is the directed angle, counterclockwise from polar axis to the segment $\overline{OP}$.

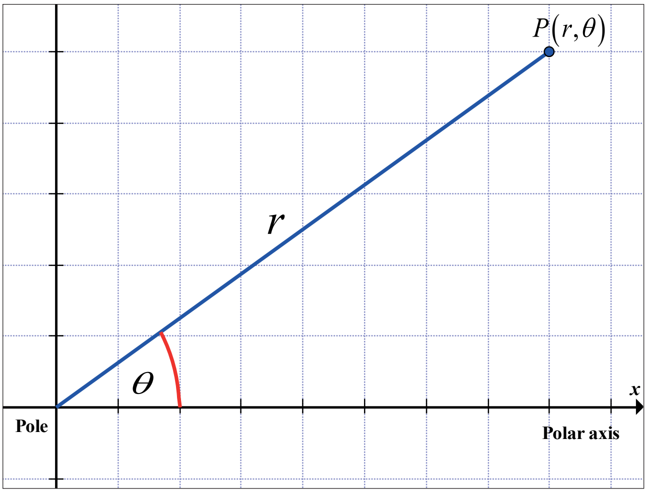


Figure 1.15. *Polar coordinates*

In polar coordinate system, the coordinates (r,θ), $(r,\theta+2k\pi)$, $k \in \mathbb{Z}$ and $(-r,\theta+(2k+1)\pi)$ represent the same point. Moreover, the pole is represented by $(0,\theta)$ where θ is any angle.

Coordinate conversion

To establish the relationship between polar and rectangular coordinates, let the polar axis coincide with the positive $x-axis$ and the pole coincide with the origin.

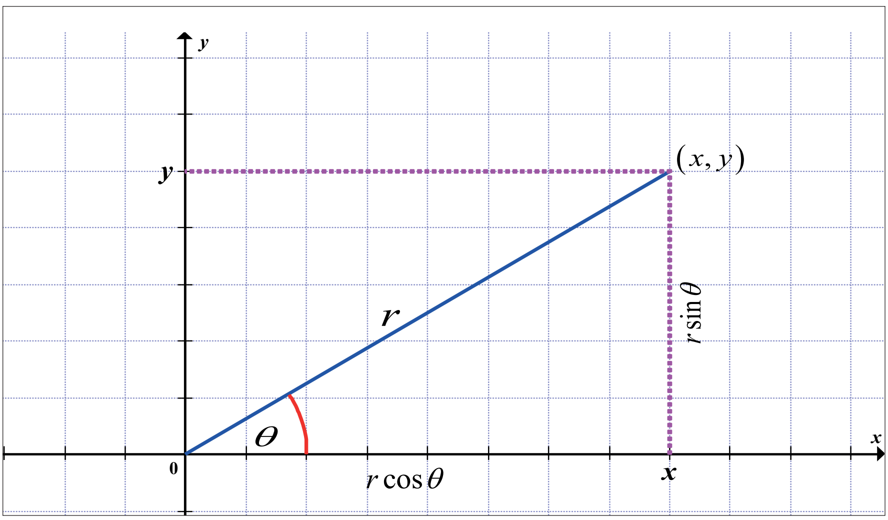


Figure 1.16. *Converting from polar coordinates to Cartesian coordinates and vice versa*

The polar coordinates (r,θ) of a point are related to the rectangular coordinates (x,y) as follows:

$$\{x = r\cos\theta, y = r\sin\theta \; and \; r = \sqrt{x^2+y^2}$$

Notice

To convert rectangular coordinates (a,b) to polar coordinates is the same as finding the modulus and argument of complex number $z = a+bi$.

Example 8.16

Given the polar coordinates $\left(2,\frac{3\pi}{4}\right)$. Find their corresponding rectangular coordinates.

Solution

In this case, we have; $r=2,\ \theta=\frac{3\pi}{4}$

$$\Rightarrow x = r\cos\theta = 2\cos\frac{3\pi}{4} = -\sqrt{2} \;\; and \;\; y = r\sin\theta = 2\sin\frac{3\pi}{4} = \sqrt{2}$$

Then the corresponding rectangular coordinates are $\left(-\sqrt{2},\sqrt{2}\right)$.

Example 8.17

Find the polar coordinates of the point $\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$

Solution

$x = \frac{\sqrt{3}}{2}$, $y = -\frac{1}{2}$ and $r = \sqrt{x^2+y^2} = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1$

Thus,

$$\cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \pm\frac{\pi}{6}$$

$$\sin\theta = -\frac{1}{2} \Rightarrow \theta = \begin{cases} -\frac{\pi}{6} \\ \frac{5\pi}{6} \end{cases}$$

The common angle is $-\frac{\pi}{6}$

Therefore, the polar coordinates are $\left(1, -\frac{\pi}{6}\right)$

Example 8.18

Express:

a) $x^2 + xy = 3$ in polar form.

b) $r = 3 - \sin\theta$ in cartesian form.

Solution

a) $x^2 + xy = 3$

Substituting $x = r\cos\theta$ and $y = r\sin\theta$ gives

$r^2\cos^2\theta + r^2\cos\theta\sin\theta = 3 \Rightarrow r^2\cos\theta(\cos\theta + \sin\theta) = 3$

b) $r = 3 - \sin\theta$

In order to be able to use $r^2 = x^2 + y^2$ and $y = r\sin\theta$, we first multiply the polar equation by r:

$$r^2 = 3r - r\sin\theta$$

$$x^2 + y^2 = 3\sqrt{x^2 + y^2} - y$$

$\Rightarrow \left(x^2 + y^2 + y\right)^2 = 9\left(x^2 + y^2\right)$ which is the required cartesian form.

Example 8.19

Illustrate graphically the curve given by polar equation $r = 2 + 2\sin t$.

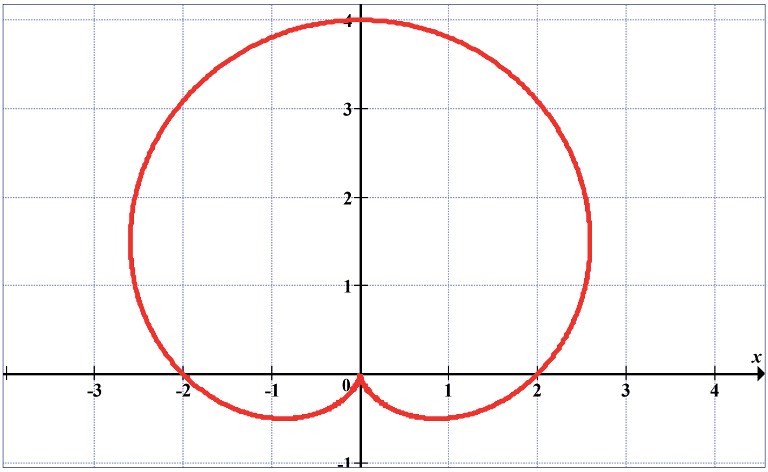
Solution

We first construct a table of values using the special angles and their multiples. r is maximum and equal to 4 for $t = \frac{\pi}{2}$. r is minimum and equal to zero when $t = \frac{3\pi}{2}$.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	2	3	3.4	3.7	4	3.7	3.4	3	2

t	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
r	1	0.6	0.3	0	0.3	0.6	1	2

We now plot the points in the table, then join them with a smooth curve. The points and the graph of the given polar equation are shown below.



Example 8.20

Graph the polar equation given by $r = 4\cos 2t$.

Solution

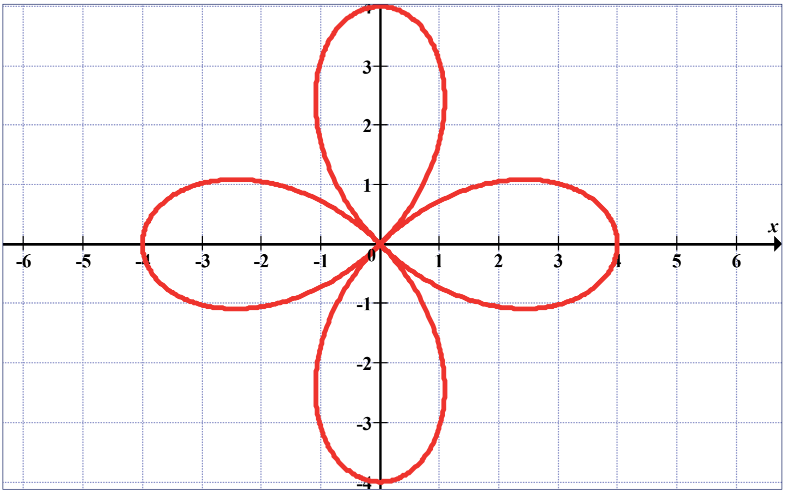
Just like in the previous example 8.18, we first construct a table of values using the special angles and their multiples. r is maximum and equal to 4 for $t = 0$ and $t = \pi$.

r is minimum and equal to -4 for $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	4	2	0	-2	-4	-2	0	2	4

t	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
r	2	0	-2	-4	-2	0	2	4

We now first plot the points in the table then join them with a smooth curve.



Application activity 8.8

1. Which of the following polar coordinates describe the same point?

   a) $(3,0)$ b) $(-3,0)$ c) $\left(2,\dfrac{2\pi}{3}\right)$

   d) $\left(2,\dfrac{7\pi}{3}\right)$ e) $(-3,\pi)$ f) $\left(2,\dfrac{\pi}{3}\right)$

   g) $(-3,2\pi)$ h) $\left(-2,-\dfrac{\pi}{3}\right)$

2. Plot the following points (given in polar coordinates).

   a) $(2,0)$ b) $(-2,0)$

   c) $\left(2,\dfrac{\pi}{2}\right)$ d) $\left(-2,\dfrac{\pi}{2}\right)$

3. Find the Cartesian coordinates of each of the points in question 1).

4. Express in Cartesian coordinates of the following polar equations.

   a) $r\cos\theta + r\sin\theta = 1$ b) $r\cos\theta = 3$

   c) $\theta = \dfrac{\pi}{4}$ d) $r = \dfrac{3}{\cos\theta - 3\sin\theta}$

   e) $r = 9$ f) $25r^2\cos^2\theta + 16r^2\sin^2\theta = 400$

   g) $r = 3\sin\theta$ h) $r^2 = 6r\cos\theta - 2r\sin\theta - 6$

   i) $r\sin^2\theta = 3\cos\theta$

8.6.2. Polar equation of a conic

Activity 2.2

1. Express the equation of the parabola $y^2 = 1+2x$ in polar equation.

2. a) Express the polar equation $r = \dfrac{6}{2+\cos\theta}$ in Cartesian equation.
   b) What are the characteristics of the equation found? Hence, identify the nature of its curve.

Another alternative way to define a conic is using polar coordinates. In polar equation of a conic, the pole is the focus of the conic.

In this case, we use the following relations:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}, \; x \neq 0$$

A conic curve with eccentricity e, focus at the origin, whose directrix $x = -p$ has equation $r = \dfrac{ep}{1+e\cos\theta}$ where (r,θ) are polar coordinates of any point P lying on the conic and $x = p > 0$ is the vertical directrix.

It is an **ellipse** if $e<1$, a **parabola** if $e=1$ and a **hyperbola** if $e>1$.

Example 8.21

Find the polar equation of an ellipse whose centre is at $(3,0)$, horizontal major axis with $10\,units$ and vertical minor axis with $8\,units$.

Solution

In the equation $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, $h=3,\ k=0$

$2a = 10 \Rightarrow a = 5 \text{ and } 2b = 8 \Rightarrow b = 4$

Then,

$$\frac{(x-3)^2}{5^2}+\frac{(y-0)^2}{4^2}=1$$

Expanding, we get $16x^2+25y^2-96x=256$

$$\Leftrightarrow 16x^2+25y^2=96x+256$$

Creating the perfect square on the right side

$$\Leftrightarrow 16x^2+9x^2+25y^2=9x^2+96x+256$$

$$\Leftrightarrow 16x^2+9x^2+25y^2=9x^2+2(48x)+256$$

$$\Leftrightarrow 25x^2+25y^2=(3x+16)^2 \Rightarrow 25\left(x^2+y^2\right)=(3x+16)^2$$

But $x^2+y^2=r^2$, $x=r\cos\theta$

Then,

$$25\left(x^2+y^2\right)=(3x+16)^2 \Rightarrow 25r^2=(3r\cos\theta+16)^2$$

$$\Rightarrow \sqrt{25r^2}=\sqrt{(3r\cos\theta+16)^2} \Rightarrow 5r=3r\cos\theta+16$$

$$\Rightarrow 5r-3r\cos\theta=16 \Rightarrow r(5-3\cos\theta)=16 \Rightarrow r=\frac{16}{5-3\cos\theta}$$

Application activity 8.9

1. Find the polar equation of the conic section:

   a) $y^2=1-2x$ b) $x^2-3y^2-8y=4$

2. Determine the cartesian equation of each of the following polar equations:

   a) $r=\dfrac{2}{2-\cos\theta}$ b) $r=\dfrac{2}{1+\sin\theta}$

3. Show that the equations $x=r\cos\theta$, $y=r\sin\theta$ transform to the polar equation $r=\dfrac{k}{1+e\cos\theta}$ and

   Cartesian equation $\left(1-e^2\right)x^2+y^2+2kex-k^2=0$.

8.6.3. Polar equation of a straight line

Activity 7.1

Consider a straight line with equation $3x-2y+6=0$.

From polar coordinates, derive the expression for $\frac{1}{r}$ which is the polar equation of the line.

From activity 8.10,considering the straight line $ax+by+c=0$, the polar equation of the straight line is $\frac{1}{r}=A\cos\theta+B\sin\theta,\ \ A,B\in\mathbb{R}$ and A and B are not all zero.

Example 8.22

Find the polar equation of the line passing through point $\left(1,\frac{\pi}{2}\right)$ and $(2,\pi)$

Solution

From; $\frac{1}{r}=A\cos\theta+B\sin\theta$

For point $\left(1,\frac{\pi}{2}\right)$; $\frac{1}{1}=A\cos\frac{\pi}{2}+B\sin\frac{\pi}{2}\Rightarrow B=1$

For point $(2,\pi)$; $\frac{1}{2}=A\cos\pi+B\sin\pi\Rightarrow A=-\frac{1}{2}$

Then, the polar equation is

$$\frac{1}{r}=-\frac{1}{2}\cos\theta+\sin\theta$$

Application activity 8.10

Determine the polar equation of each of the following lines:

1. $x+\sqrt{3}\,y=4$
2. $x-y=2$
3. $y=\frac{\sqrt{3}}{2}(x-1)$
4. $x-2y=\sqrt{5}$

8.6.4. Polar equation of a circle

Activity 2.2

Consider the following figure:

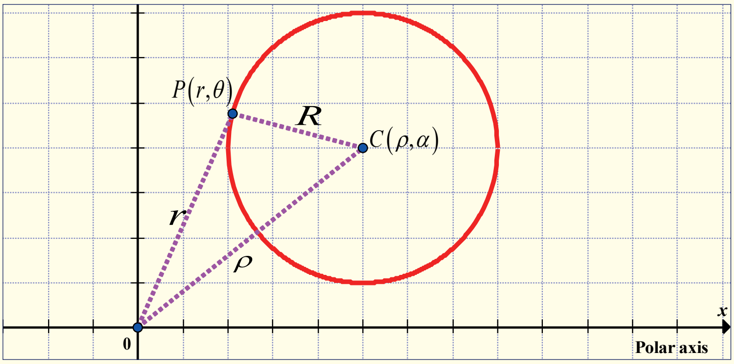
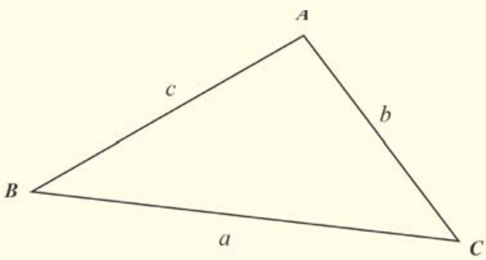


Figure 1.17. *Circle in polar coordiantes*

Hint:

Consider the following triangle



The cosine law states that

$$\begin{cases} a^2 = b^2 + c^2 - 2bc\cos\hat{A} \\ b^2 = a^2 + c^2 - 2ac\cos\hat{B} \\ c^2 = a^2 + b^2 - 2ab\cos\hat{C} \end{cases}$$

Using cosine law, derive the expression for r^2

From activity 8.11, the polar equation of a circle with centre (ρ,α) and radius R is

$$r^2 = R^2 - \rho^2 + 2r\rho\cos(\theta-\alpha)$$

Example 8.23

Find the polar equation of the circle with centre $\left(3,-\frac{\pi}{6}\right)$ and radius 2.

Solution

In the equation $r^2 = R^2 - \rho^2 + 2r\rho\cos(\theta-\alpha)$, $R=2,\ \rho=3,\ \alpha=-\frac{\pi}{6}$

The equation is

$$r^2 = 4-9+6r\cos\left(\theta+\frac{\pi}{6}\right)$$

$$\Rightarrow r^2 = -5+6r\cos\left(\theta+\frac{\pi}{6}\right)$$

Application activity 8.11

Determine the polar equation of the circle:

1. Whose radius 3 and centre $(3,0)$.
2. Whose radius 2 and centre $\left(2,\frac{\pi}{2}\right)$.
3. Whose radius $\frac{1}{2}$ and centre $\left(-\frac{1}{2},0\right)$.
4. Whose radius 1 and centre $\left(-1,\frac{\pi}{2}\right)$.

8.6. Applications

Activity 2.3

Is the Earth a perfect sphere? Justify your answer by giving facts.

Eccentricities of orbits of the planets

The orbits of planets are ellipses with the sun at one focus. For most planets, these ellipses have very small eccentricity, so they are nearly circular. However, Mercury and Pluto, the innermost and outermost known planets respectively, have visibly elliptical orbits.

Planets	Eccentricity
Mercury	0.206
Venus	0.007
Earth	0.017
Mars	0.093
Jupiter	0.048
Saturn	0.056
Uranus	0.046
Neptune	0.010
Pluto	0.248

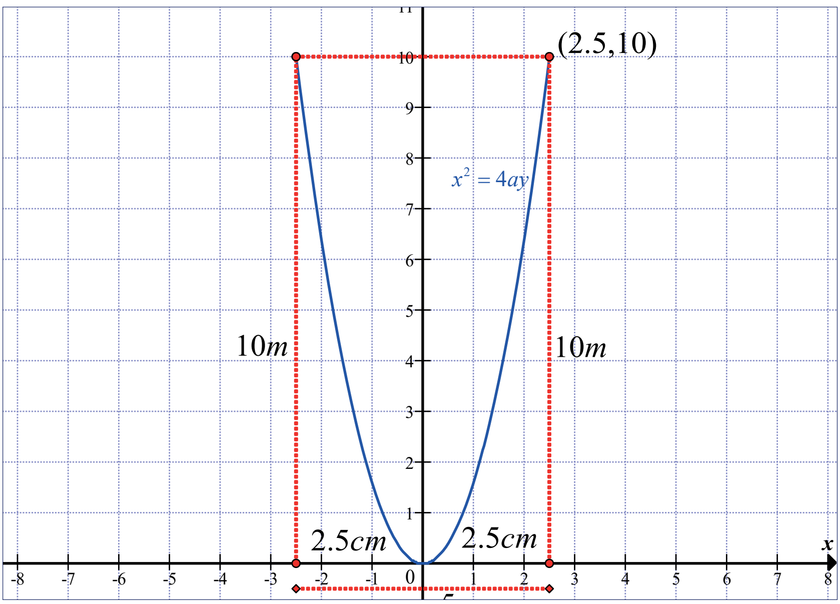
The following examples illustrate many of the practical applications of conics.

Example 8.24

An arch is in the form of a parabola with its axis vertical. The arch is $10\,m$ high and $5\,m$ wide at the base. How wide is it $2\,m$ from the vertex of the parabola?

Solution

It is given that arch is in the parabolic form with its vertical axis.



Let the vertex of the parabola be at the origin and the axis be $y-axis$;

Thus, the equation of the parabola is of the form $x^2 = 4ay$.

Since $(2.5,10)$ lies on the parabola, then $(2.5)^2 = 4a(10) \Rightarrow 6.25 = 40a$ $\Rightarrow a = \dfrac{625}{4000} = \dfrac{5}{32}$.

Therefore, the equation of parabolic arch is $x^2 = 4\times\dfrac{5}{32}y$ or $x^2 = \dfrac{5}{8}y$.

When $y = 2,\ x^2 = \dfrac{5}{8}\times 2 = \dfrac{5}{4} \Rightarrow x = \dfrac{\sqrt{5}}{2}$.

The width of the arch at height of $2m$ from the vertex is

$2\times\dfrac{\sqrt{5}}{2}m = \sqrt{5}\,m$.

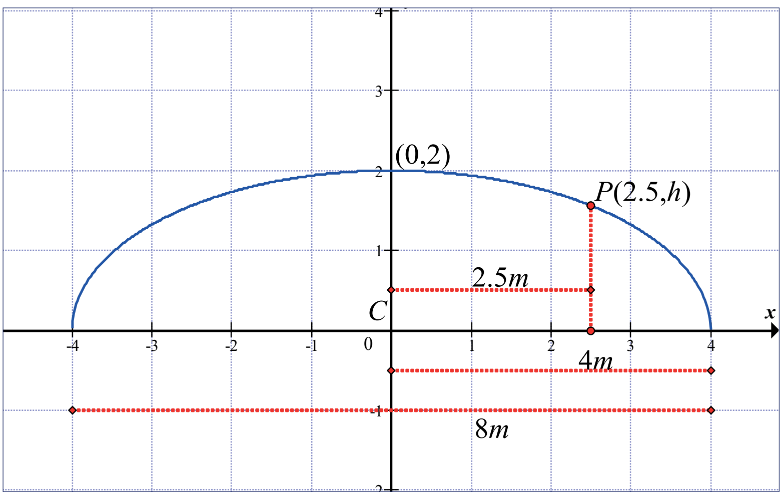
Example 8.25

An arch is in the form of a semi-ellipse. It is $8m$ wide and $2m$ high at the centre. Find the height of the arch at a point $1.5\ m$ from one end.

Solution

Let the $x-axis$ lie along the base of arch, with the origin at the middle of the base.

Let the equation of the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.



Let the vertex of the parabola be at the origin and the axis be $y-axis$; that is, equation of the parabola is of the form $x^2 = 4ay$.

Since $2a = 8 \Rightarrow a = 4$.

The point $(0,2)$ lies on the ellipse, then $\frac{(0)^2}{a^2} + \frac{(2)^2}{b^2} = 1$ or $\frac{4}{b^2} = 1 \Rightarrow b^2 = 4$;

The point $(2.5, h)$ also lies on our ellipse, $\Rightarrow \frac{(2.5)^2}{a^2} + \frac{(h)^2}{b^2} = 1$

$\Leftrightarrow \frac{6.25}{16} + \frac{h^2}{4} = 1,$ with $s\, a = 4\, and\, b^2 = 4$

$$\Rightarrow \frac{h^2}{4} = 1 - \frac{6.25}{16} = \frac{975}{1600} \Rightarrow h^2 = \frac{975}{1600} \times 4 = 2.4375$$

$$\Rightarrow h = \sqrt{2.4375} = 1.56$$

Hence, the height of the arch at a point $1.5\ m$ from one end is $1.56\,m$.

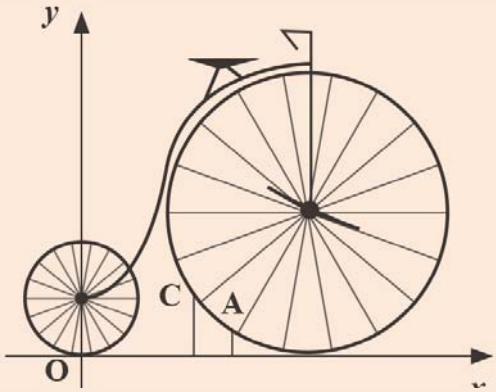
Application activity 8.12

1. The cross-section of a reflector of a torch is modelled by the part of $y = 6 - 0.24x^2$ which lies above the x-axis, where x and y are both measured in cm. Draw this curve and find the;

   a) depth of the reflector,

   b) diameter of the mouth of the reflector.

2. A penny-farthing bicycle on display in a museum is supported by a stand at points A and C.
   A and C lie on the front wheel.



With coordinate axes as shown and $1unit = 5cm$,
the equation of rear wheel (the small wheel) is

$x^2 + y^2 - 6y = 0$ and the equation of the front wheel is
$x^2 + y^2 - 28x - 20y + 196 = 0$.

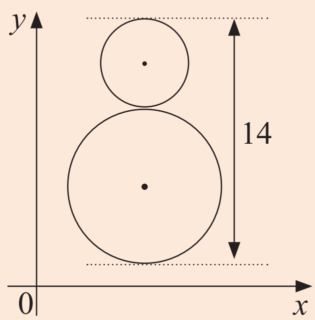
a) i) Find the distance between the centres of the two wheels.

ii) Hence calculate the clearance, i.e. the smallest gap between the front and rear wheels. Give your answer to the nearest millimeter.

b) i) $B(7,3)$ is half-way between A and C where P is the centre of the front wheel. Find the gradient of PB.

ii) Hence, find equation of AC and the coordinates of A and C.

3. A bakery firm makes gingerbread men each $14\,cm$ high with circular heads and bodies.



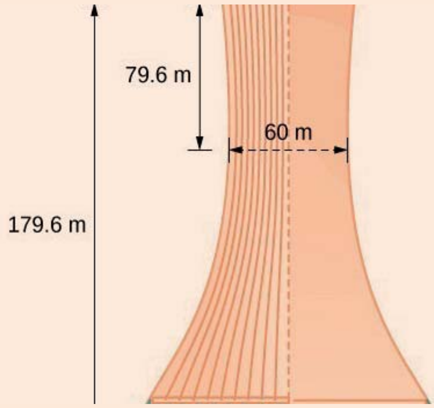
The equation of body is $^2 + y^2 - 10x - 12y + 45 =$ and the line of centres is parallel to the y -axis. Find the equation of head.

4. An arc is in the form of a parabola with its axis vertical. The arc is $10\,m$ high and $5\,m$ wide at the base. How wide is it $2\,m$ from the vertex of the parabola?

5. Satellites can be put into elliptical orbits if they need only sometimes to be in high- or low-earth orbit, thus avoiding the need for propulsion and navigation in low-earth orbit and the expense of launching into high-earth orbit.
Suppose a satellite is in an elliptical orbit, with $b = 4416$ and with the centre of the Earth being at one of the foci of the ellipse.

Assuming the Earth has a radius of about 3960 miles, find the lowest and highest altitudes of the satellite above the Earth.

6. The design layout of a cooling tower is shown in figure below. The tower stands 179.6 metres tall. The diameter of the top is 72 metres. At their closest, the sides of the tower are 60 metres apart.



Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the centre of the hyperbola indicated by the intersection of dashed perpendicular lines in the above figure is the origin of the coordinate plane. Round off final values to four decimal places.

7. A whispering room is one with an elliptically-arched ceiling. If someone stands at one focus of the ellipse and whispers something to his friend, the dispersed sound waves are reflected by the ceiling and concentrated at the other focus, allowing people across the room to clearly hear what the person said. Suppose such gallery has a ceiling reaching twenty feet above the five-foot-high vertical walls at its tallest point (so the cross-section is half an ellipse topping two vertical lines at either end), and suppose the foci of the ellipse are thirty feet apart. What is the equation for the elliptical ceiling and the height of the ceiling above each whispering point?

Unit summary

1. Generalities on conic sections

Parabolas, circles, ellipses and hyperbolas are called conics because they are curves in which planes intersect right circular cones.

2. Parabola

A parabola is the set of all points in plane that are equidistant from a fixed line (called directrix) and a fixed point (called focus) not on the line.

Important result relating to different parabolas

Equation	$y^2 = 4ax$	$x^2 = 4ay$
Focus	$(a,0)$	$(0,a)$
Directrix	$x = -a$	$y = -a$
Principal axis(the line through the focus perpendicular to the directrix)	$y = 0$	$x = 0$
Vertex (point where the parabola crosses its principal axis)	$(0,0)$	$(0,0)$
Length of latus rectum (length of chord through a focus and perpendicular to the principal axis)	$4a$	$4a$
Equation of latus rectum	$x = a$	$y = a$
Ends of latus rectum	$(a, \pm 2a)$	$(\pm 2a, a)$

Replacing x with $x - h$ has the effect of shifting the graph of an equation by $|h|$ units to the right if h is positive, to the left if h is negative.

Similarly, replacing y with $y - k$ has the effect of shifting the graph by $|k|$ units up if k is positive and down if k is negative.

Equation	$(y-k)^2 = 4p(x-h)$	$(x-h)^2 = 4p(y-k)$
Focus	$(\mathrm{h}+p, k)$	$(h, k+p)$
Directrix	$x = h - p$	$y = k - p$
Principal axis(the line through the focus perpendicular to the directrix)	$y = k$	$x = h$
Vertex (point where the parabola crosses its principal axis)	(h,k)	(h,k)

Parametric equations of parabola are

$$\begin{cases} x = at^2 \\ y = 2at \end{cases}$$ where t is a parameter.

The tangent line at point x_0, y_0, on parabola $y^2 = 4ax$, is given by

$$T \equiv y_0 y = 2a(x + x_0)$$

3. Ellipse

Ellipse is a set of all points in the plane, the sum of whose distances from two fixed points (called foci) is a given positive constant.

Important facts to different ellipses

Equation of Standard form	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b > 0$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ 0 < a < b$
Coordinates of centre	$(0,0)$	$(0,0)$
Coordinates of vertices	$(a,0)$ and $(-a,0)$	$(0,b)$ and $(0,-b)$
Length of major axis	$2a$	$2b$
Equation of major axis	$y = 0$	$x = 0$
Length of minor axis	$2b$	$2a$
Equation of minor axis	$x = 0$	$y = 0$
Eccentricity (ratio of semi-focal separation and the semi-major axis)	$b^2 = a^2(1-e^2) \Rightarrow e = \frac{\sqrt{a^2 - b^2}}{a}$	$a^2 = b^2(1-e^2) \Rightarrow e = \frac{\sqrt{b^2 - a^2}}{b}$
Coordinates of foci	$(ae,0)$ and $(-ae,0)$ $\Leftrightarrow \left(\pm\sqrt{a^2 - b^2}, 0\right)$	$(0,be)$ and $(0,-be)$ $\Leftrightarrow \left(0, \pm\sqrt{b^2 - a^2}\right)$

Equation of directrices	$x = \pm\frac{a}{e}$	$y = \pm\frac{b}{e}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Equations of latus rectum	$x = \pm ae$	$y = \pm be$

Parametric equations of ellipse with centre (h,k) are

$\begin{cases} x = h + a\cos t \\ y = k + b\sin t \end{cases}$ where t is a parameter and $t \in (-\pi, \pi]$

The **tangent line** at point (x_0, y_0), on ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is given by

$$T \equiv \frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

4. Hyperbola

Hyperbola is a set of all points in the plane, the difference of whose distances from two fixed points (foci) is a given positive constant

Important facts to different hyperbolas

Equation of Standard form	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$
Coordinates of centre	$(0,0)$	$(0,0)$
Coordinates of vertices	$(a,0)$ and $(-a,0)$	$(0,b)$ and $(0,-b)$
Length of transverse axis	$2a$	$2b$
Equation of transverse axis	$y = 0$	$x = 0$
Equation of conjugate axis	$x = 0$	$y = 0$
Coordinates of foci	$(ae,0)$ and $(-ae,0)$ $\Leftrightarrow \left(\pm\sqrt{a^2+b^2}, 0\right)$	$(0,be)$ and $(0,-be)$ $\Leftrightarrow \left(0, \pm\sqrt{a^2+b^2}\right)$

Equation of directrices	$x = \pm\frac{a}{e}$	$y = \pm\frac{b}{e}$
Length of latus rectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Equations of latus rectum	$x = \pm ae$	$y = \pm be$
Eccentricity	$b^2 = a^2\left(1-e^2\right)$	$a^2 = b^2\left(1-e^2\right)$

Parametric equations of hyperbola are

$\begin{cases} x = a\sec t \\ y = b\tan t \end{cases}$ where t is a parameter and

$t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\cup \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[$

The tangent line at point $\left(x_0, y_0\right)$, on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,is given by

$$T \equiv \frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$$

5. Polar coordinates

To form a polar coordinate system in the plane, we fix a point 0 called the pole (or origin) and construct from 0 an initial ray called the polar axis. Then each point P in the plane can be assigned polar coordinates (r, θ) as follows:

- r is the directed distance from 0 to P.
- θ is the directed angle, counterclockwise from polar axis to the segment $\overline{OP}$.

In polar coordinate system, the coordinates (r,θ),$(r,\theta+2k\pi)$, $k \in \mathbb{Z}$ and $(-r, \theta+(2k+1)\pi)$ represent the same point.

Coordinate conversion

The polar coordinates (r,θ) of a point are related to the rectangular coordinates (x, y) as follows:

$x = r\cos\theta,\quad y = r\sin\theta,\quad r^2 = x^2 + y^2,\quad \tan\theta = \frac{y}{x},\ x \neq 0$

Polar equation of a conic

A conic curve with eccentricity e, focus at the origin, whose directrix $x = -p$ has equation $r = \frac{ep}{1 + e\cos\theta}$
where (r,θ) are polar coordinates of any point P lying on the conic.

It is an **ellipse** if $e < 1$, a **parabola** if $e = 1$, a **hyperbola** if $e > 1$.

6. Applications

Eccentricities of orbits of the planets

The orbits of planets are ellipses with the sun at one focus. For most planets, these ellipses have very small eccentricity, so they are nearly circular. However, the Mercury and Pluto, the innermost and outermost known planets respectively, have visibly elliptical orbits.

End of unit assessment

1. Describe the conic having the given equations. Give its foci and principal axes and, if it is a hyperbola, its asymptotes:

   a) $x^2 = -8y$ b) $x^2 + 2y^2 = 2$ c) $x = -3y^2$

   d) $y^2 - x^2 = 8$ e) $x + y^2 = 2y + 3$ f) $8x^2 - 2y^2 = 16$

2. For each of the following, find all intersections of the given curves, and make a sketch of the curves that show the points of intersection:

   a) $x^2 - 4y^2 = 36\ and\ x - 2y - 20 = 0$

   b) $y^2 - 8x^2 = 5\ and\ y - 2x^2 = 0$

   c) $3x^2 - 7y^2 = 5\ and\ 9y^2 - 2x^2 = 1$

   d) $x^2 - y^2 = 1\ and\ y^2 + x^2 = 7$

3. Find equation of ellipse traced by a point that moves so that the difference between its distances to $(4,1)$ and $(4,5)$ is 12.

4. Find the equation of the hyperbola traced by a point that moves so that the sum of its distances to the points $(0,0)$ and $(1,1)$ is 1.

5. Let $4x^2-4xy+y^2+6x+1=0$ be equation of a conic. Determine the values of k for which the line $y=kx$;

   a) intersects the given conics once,

   b) cuts the given conics in two points,

   c) does not intersect the given conics.

6. What points in the xy $plane$ satisfy the equations and inequalities in the following curves? In each case, illustrate graphically.

   a) $\left(x^2-y^2-1\right)\left(x^2+y^2-25\right)\left(x^2+4y^2-4\right)=0$

   b) $\left(x+y\right)\left(x^2+y^2-1\right)=0$

   c) $\dfrac{x^2}{9}+\dfrac{y^2}{16}\leq 1$

   d) $\dfrac{x^2}{9}-\dfrac{y^2}{16}\leq 1$

   e) $\left(9x^2+4y^2-36\right)\left(4x^2+9y^2-16\right)\leq 0$

   f) $\left(9x^2+4y^2-36\right)\left(4x^2+9y^2-16\right)>0$

7. For each of the following equations, it is given how many units up or down and to the right or left each conic is to be shifted.

   Find an equation for the new conic, and indicate the new vertex, focus and directrix for parabola; the new foci, vertices, centre and asymptotes if any.

   a) $y^2=4x$, left 2,down 3

   b) $x^2=8y$, right 1, down 7

   c) $\dfrac{x^2}{6}+\dfrac{y^2}{9}=1$, left2, down 1

   d) $\dfrac{x^2}{3}+\dfrac{y^2}{2}=1$, right 2, up 3

e) $\frac{x^2}{4}-\frac{y^2}{5}=1$, right 2, up2

f) $y^2-x^2=1$, left 1, down1.

8. Find rectangular coordinates of the points whose polar coordinates are given:

a) $\left(6,\frac{\pi}{6}\right)$ b) $\left(7,\frac{2\pi}{3}\right)$ c) $\left(8,\frac{9\pi}{4}\right)$

d) $(5,0)$ e) $\left(7,\frac{17\pi}{6}\right)$ f) $(0,\pi)$

9. The following are rectangular coordinates.

a) $(-5,0)$ b) $\left(2\sqrt{3},-2\right)$ c) $(0,-2)$

d) $(-8,-8)$ e) $\left(-3,3\sqrt{3}\right)$ f) $(1,1)$

Express the points in polar coordinates, with

(i) $r\geq 0$ and $0\leq\theta<2\pi$

(ii) $r\leq 0$ and $0\leq\theta<2\pi$

10. In each of the following, transform the given polar equation to rectangular coordinates and identify the curve represented.

a) $r=5$ b) $r\sin\theta=4$ c) $r=\frac{1}{1-\cos\theta}$

d) $r=\frac{2}{1-2\sin\theta}$ e) $r=\frac{5}{3\sin\theta-4\cos\theta}$

f) $r=\frac{6}{2-\cos\theta}$ g) $r+4\cos\theta=0$

11. A planet travels about its sun in an ellipse whose semi-major axis length a.

a) Show that $r=a(1-e)$ when the planet is closest to the sun (perihelion) and $r=a(1+e)$ when the planet is farthest from the sun (aphelion).

b) Use the data in the table below to find how close each planet in our solar system comes to the sun and how far away each planet gets from the sun.

Planets	Semi-major axis (astronomical units)	Eccentricity
Mercury	0.3871	0.2056
Venus	0.7233	0.0068
Earth	1.000	0.0167
Mars	1.524	0.0934
Jupiter	5.203	0.0484
Saturn	9.539	0.0543
Uranus	19.18	0.0460
Neptune	0.0082	0.0082

c) Use the data from the table above to find polar equations for the orbits of the planets

12. A man running a race-course discovers that the sum of the distances from the two flag posts from him is always $10m$ and the distance between the flag post is $8m$. Find the equation of the path traced by the man.

13. The towers of a bridge, hung in the form of a parabola, have their tops $30m$ above the roadway and $200m$ apart. If the cable is $5m$ above the roadway at the centre of the bridge, find the length of the vertical supporting cable $30m$ from the centre.

14. Given two points A and B where $\overline{AB}=6$. Find in its simplest form the equation of the locus of point which moves such that $\overline{PA}+\overline{PB}=8$.

15. Ellis built a window frame shaped like the top half of an ellipse. The window is 40 inches tall at its highest point and 160 inches wide at the bottom. What is the height of the window 20 inches from the centre of the base?

16. A forest ranger at an outpost in the Sam Houston National Forest and another ranger at the primary station both heard an explosion. The outpost and the primary station are 6 km apart. If one ranger heard the explosion 6 s before the other, write an equation that describes all the possible locations of the explosion. Locate the two ranger stations on the x-axis with the midpoint between the stations at the origin. The transverse axis is horizontal. (**Hint**: The speed of sound is about 0.35 kilometre per second).

Unit 9

Random Variables

Introductory activity

A bag contains 6 blue pens and 4 red pens. Three pens are drawn and not replaced. If **x** stands for the number of blue pens and **y** the number of red pens drawn

Complete the following table to illustrate different situations you can have.

x	y	Total number =3
0	3	3
1	...	...
2	...	...
3	...	...

Can you determine exact number of blue pens to be selected at any time if you do not know the number of red pens selected?

Does this number remain the same? Explain you answer.

Objectives

By the end of this unit, a student will be able to:

- Define a random variable
- Identify whether a given random variable is discrete or continuous.
- Define the parameters of a discrete random variable.
- Learn in which situation the Binomial distribution applies and state its parameters, …

9.1. Discrete and finite random variables

9.1.1. Probability density function

Activity 2.4

Suppose a box contains 6 balls of which 4 are red and 2 are black. Three balls are withdrawn one after the other. Let x represent the number of red balls drawn. Construct a table to represent this probability distribution if each ball is replaced before another is withdrawn. What can you say about the value x takes on the sum of obtained probabilities?

Suppose that the outcome set S of an experiment is divided into n mutually exclusive and exhaustive events $E_1, E_2, E_3, ..., E_n$.

A variable X which can assume numerical values each of which can correspond to one and only one of the events is called a **random variable** because outcomes depend on chance..

A random variable X is said to be a **discrete random variable**, if it takes only finite values between its limits; for example, the number of students appearing in a festival consisting of 400 students is a discrete random variable which can assume values other than 0, 1, 2, …,400.

Discrete random variables are usually (but not necessarily) countable. Their values can be finite or countably infinite.

Random variables are usually denoted by upper case letters. The possible values a random variable assumes are denoted by the corresponding lower case letters and thus we write $X = x$.

Before talking about probability density function, let us remember some key words of probability theory.

The **sample space** corresponds to the set of all possible outcomes of the experiment. Elements of a sample space are called **outcomes**.

An **event** is a subset of a sample space.

Distribution function for a random variable X is a real function M whose domain is Ω.

The random variables are described by their probabilities. i.e

$$P(X = x_1) = p_1,\ P(X = x_2) = p_2,\ ...,\ P(X = x_n) = p_n.$$

The distribution of probabilities $P(X = x_i) = p_i$ is called the **probability distribution** and satisfy

- $M(w) \geq 0$ for all $w \in \Omega$
- $\sum_i M(w_i) = 1$

Then, for any subset E of Ω, the probability of E is the number $p(E) = \sum_{w \in E} M(w) = p(X = x_i)$.

Then X is called a discrete random variable if $\sum_{i=1}^{n} p_i = 1$.

The **probability density function** (p.d.f), $f(x)$, is a function that allocates probabilities to all distinct values that X can take on.

Notice

If the initial probability is known, you can find successive probabilities using the following recurrence relation

$$P(X = x+1) = \left(\frac{n-x}{x+1}\right)\left(\frac{p}{q}\right) P(X = x)$$

In fact, $P(X = x) = {}^nC_x p^x q^{n-x} \quad (1)$

and

$$P(X = x+1) = {}^nC_{x+1} p^{x+1} q^{n-x-1} \quad (2)$$

Dividing (2) by (1) yields

$$\frac{P(X = x+1)}{P(X = x)} = \frac{{}^nC_{x+1} p^{x+1} q^{n-x-1}}{{}^nC_x p^x q^{n-x}} = \frac{n-x}{x+1} \cdot \frac{p}{q}$$

Thus,

$$P(X = x+1) = \left(\frac{n-x}{x+1}\right)\left(\frac{p}{q}\right) P(X = x)$$

Example 9.1

A game involves throwing a six sided die. Find;

a) The outcome of X is even

b) $P(X < 1)$ c) $P(1 < X \leq 2)$ d) $P(2 \leq X \leq 3)$

e) $P(1 \le X < 6)$ f) $P(X \le 6)$

g) $p(E)$ where $E = X$ is an even.

Solution

In this case, the possible outcomes are 1,2,3,4,5,6.

P (any number to appear) $\frac{1}{6}$

Let X be the random variable "the number that appears"

Thus, we can write; $P(X = x_i) = \frac{1}{6}, \ x_i = 1, 2, ..., 6$

a) The sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$

b) $P(X < 1) = 0$ impossible event

c) $P(1 < X \le 2) = P(X = 2) = \frac{1}{6}$

d) $P(2 \le X \le 3) = P(X = 2) + P(X = 3)$

$$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

e) $P(1 \le X < 6) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{5}{6}$$

f) $P(X \le 6) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)$

$$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$$

g) Let event E be "the result of the roll is an even number"

Thus, $E = \{2, 4, 6\}$, and then

$$p(E) = M(2) + M(4) + M(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}.$$

Example 9.2

A bag contains 6 blue balls and 4 red balls. Three balls are drawn and not replaced. Determine the probability distribution for the number of red balls drawn.

Solution

$$\left.\begin{array}{l}6 \text{ blue balls} \\ 4 \text{ red balls}\end{array}\right\} \Rightarrow 10 \text{ balls}$$

Let B : blue ball, R : red ball

If X is the random variable "the number of red ball drawn", we have;

$$P(X=0)=P(\text{no red ball})=P(BBB)$$

$$=\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}=\frac{2}{12}=\frac{1}{6}$$

$$P(X=1)=P(1 \text{ red ball and } 2 \text{ blue balls})$$

$$=P(RBB)+P(BRB)+P(BBR)$$

$$=\frac{4}{10}\times\frac{6}{9}\times\frac{5}{8}+\frac{6}{10}\times\frac{4}{9}\times\frac{5}{8}+\frac{6}{10}\times\frac{5}{9}\times\frac{4}{8}=\frac{1}{2}$$

$$P(X=2)=P(2 \text{ red balls and } 1 \text{ blue ball})$$

$$=P(RRB)+P(RBR)+P(BRR)$$

$$=\frac{4}{10}\times\frac{3}{9}\times\frac{6}{8}+\frac{4}{10}\times\frac{6}{9}\times\frac{3}{8}+\frac{6}{10}\times\frac{4}{9}\times\frac{5}{8}=\frac{3}{10}$$

$$P(X=3)=P(\text{no blue ball})=P(RRR)$$

$$=\frac{4}{10}\times\frac{3}{9}\times\frac{2}{8}=\frac{1}{30}$$

Thus, we have;

x	0	1	2	3
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

Application activity 9.13

1. A discrete variable X has probability distribution defined by $P(X=x)=\frac{1}{6}(x-1)$, for $x=2,3,4$. Show that X is a random variable.

2. A discrete random variable has the following probability distribution:

x	1	2	3	4
$P(X=x)$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	p

Find; a) the value of p.

b) $P(X \geq 2)$.

3. The probability distribution of a discrete random variable T is given by $P(T=t)=a\left(\frac{2}{3}\right)^{t}$, for $t=1,2,3,\ldots$. Find the value of a.

Cumulative distribution of discrete random variable

Activity 2.5

Recall, in statistics, that Cumulative frequency can be defined as the sum of all previous frequencies up to the current point. Use this fact to complete the table below, for the number of heads obtained when an unbiased coin with sides labeled head (H) and tail (T) is tossed four times.

Heads	0	1	2	3	4
Probability	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$
Cumulative Probability	…	…	…	…	…

To find a **cumulative probability,** we add the probabilities for all values qualifying as “less than or equal” to the specified value.

The **cumulative distribution** function of a random variable X is the function $F(x)=P(X \leq x)$.

Example 9.3

Suppose that the range of a discrete random variable is $\{0,1,2,3,4\}$ and its probability density function is $f(x)=\dfrac{x}{10}$. What is its cumulative distribution function?

Solution

For

$x<1,\ F(x)=f(0)=0$

$1\le x<2,\ F(x)=f(0)+f(1)=0+\dfrac{1}{10}$

$2\le x<3,\ F(x)=f(0)+f(1)+f(2)=0+\dfrac{1}{10}+\dfrac{2}{10}=\dfrac{3}{10}$

$3\le x<4,\ F(x)=f(0)+f(1)+f(2)+f(3)=0+\dfrac{1}{10}+\dfrac{2}{10}+\dfrac{3}{10}=\dfrac{6}{10}$

$4\le x,\ F(x)=f(0)+f(1)+f(2)+f(3)+f(4)=0+\dfrac{1}{10}+\dfrac{2}{10}+\dfrac{3}{10}+\dfrac{4}{10}=\dfrac{10}{10}=1$

Then,

x	0	1	2	4	5
$F(x)$	0	$\dfrac{1}{10}$	$\dfrac{3}{10}$	$\dfrac{6}{10}$	1

Thus, we can write the cumulative distribution function as;

$$F(x)=\begin{cases} 0, & x<1 \\ \dfrac{1}{10}, & 1\le x<2 \\ \dfrac{3}{10}, & 2\le x<3 \\ \dfrac{6}{10}, & 3\le x<4 \\ 1, & 4\le x \end{cases}$$

Example 9.4

A discrete random variable X has the cumulative distribution

x	0	1	2	4	5
$F(x)$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{8}{10}$	1

Determine the probability distribution of X.

Solution

The cumulative distribution only changes value at $0,1,2,4,5$. So the range of X is $\{0,1,2,4,5\}$.

$F(0)=\frac{1}{10}$ so $f(0)=\frac{1}{10}$

$F(1)=f(0)+f(1)=\frac{3}{10}$ so $f(1)=\frac{3}{10}-\frac{1}{10}=\frac{2}{10}$

$F(2)=f(0)+f(1)+f(2)=\frac{5}{10}$ so $f(2)=\frac{5}{10}-\frac{1}{10}-\frac{2}{10}=\frac{2}{10}$

$F(4)=f(0)+f(1)+f(2)+f(4)=\frac{8}{10}$

So $f(4)=\frac{8}{10}-\frac{1}{10}-\frac{2}{10}-\frac{2}{10}=\frac{3}{10}$

$F(5)=f(0)+f(1)+f(2)+f(4)+f(5)=\frac{10}{10}$

So $f(5)=\frac{10}{10}-\frac{1}{10}-\frac{2}{10}-\frac{2}{10}-\frac{3}{10}=\frac{2}{10}$

Then, we write

x	0	1	2	4	5
$f(x)$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{2}{10}$

Exercise 9.14

1. I toss a coin twice. Let X be the number of observed heads. Find the cumulative distribution function of X.
2. We roll both dice at the same time and add the two numbers that are shown on the upward faces. Let X be the discrete random variable associated to this sum. Find its cumulative distribution.
3. The discrete random variable X has cumulative density function $F(x)=\frac{x}{5}$ for $x=1,2,3,4,5$. Find the probability distribution of X.

9.1.2. Expected value, variance and standard deviation

Activity 7.6

Complete the following table for a discrete random variable X.

x	$P(X=x)$	$xP(X=x)$	$x^2P(X=x)$
1	0.2	…	…
2	0.5	…	…
3	0.3	…	…
Sum	…	…	…

The **expected value** of random variable X, which is the **mean** of the probability distribution of X is denoted and defined by

$$\mu=E(X)=\sum_{i=1}^{n}x_iP(X=x_i)$$

Also, the expectation of any function $g(X)$ of the random variable X is

$$\mu=E(g(X))=\sum_{i=1}^{n}g(x)P(X=x_i)$$

The variance of random variable X is denoted and defined by

$$\sigma^2=Var(X)=\sum_{i=1}^{n}\left[x_i-E(X)\right]^2 p(X=x_i);$$

which can be simplified to

$$\sigma^2 = Var(X) = E(X^2) - [E(X)]^2$$

Since the term $\sum_{i=1}^{n} x_i^2 P(X = x_i)$ is also written as $E(X^2)$,

the standard deviation of random variable X, denoted by $\text{SD}(X)$, is the square root of the variance. That is

$$\sigma = SD(X) = \sqrt{Var(X)}.$$

Example 9.5

The following probability distribution has a random variable X.

x	0	1	2	3	4	5
$P(X = x)$	0.05	0.10	0.20	0.40	0.15	0.10

Find the;

a) expected value,

b) variance,

c) standard deviation.

Solution

a) $E(X) = \sum_{i=1}^{6} x_i P(X = x_i)$

$$= 0 \times 0.05 + 1 \times 0.10 + 2 \times 0.20 + 3 \times 0.40 + 4 \times 0.15 + 5 \times 0.10$$

$$= 2.80$$

b) $\text{var}(X) = \left[\sum_{i=1}^{n} x_i^2 P(X = x_i)\right] - [E(X)]^2$

$$= 0^2 \times 0.05 + 1^2 \times 0.10 + 2^2 \times 0.20 + 3^2 \times 0.40 + 4^2 \times 0.15 + 5^2 \times 0.10 - (2.80)^2$$

$$= 9.40 - 7.84 = 1.56$$

c) $\text{SD}(X) = \sqrt{\text{var}(X)} = \sqrt{1.56} = 1.2490$

Properties for mean and variance

$\forall a, b \in \mathbb{R}$

1. $E(a) = a$
2. $E(aX) = aE(X)$
3. $E(aX+b) = aE(X)+b$
4. $E(X+Y) = E(X)+E(Y)$
5. $E(aX^2+b) \neq aE(X^2)+b$
6. $\text{var}(a) = 0$
7. $\text{var}(aX) = a^2 \text{var}(X)$
8. $\text{var}(aX+b) = a^2 \text{var}(X)$

Application activity 3.15

1. Find the mean of each of the following discrete probability distribution:

a)

x	0	1	2	3
$P(X=x)$	0.1	0.2	0.4	0.3

b)

x	1	2	3
$P(X=x)$	0.1	0.4	0.5

c)

x	-1	0	1	2
$P(X=x)$	0.2	0.3	0.4	0.1

d)

x	4	6	8	10
$P(X=x)$	0.002	0.040	0.299	0.659

2. The random variable X has the following probability distribution:

x	2	4	6
$P(X=x)$	a	$2a^2-a$	a^2+a-1

Find the:

a) Value of a

b) $E(X)$

c) $\text{Var}(X)$

d) $\text{SD}(X)$

3. Calculate the expected value, variance, and standard deviation of the probability distribution for the possible outcomes that can be obtained by throwing a die.

9.1.3. Binomial distribution (Law of Bernoulli)

Definition

Activity 2.7

Suppose that we need to determine the probability of getting 4 heads in 10 coin tosses. In this case, $n=10$ is the number of independent trials. If getting a head is a "success (S)" then getting a tail is a "fail (F)". Therefore, the number of successes is $r=4$ and the number of fails is $n-r=10-4=6$ in 10 trials. Here, if the first 4 tosses are heads, the last 6 are tails. That is $SSSSFFFFFF$.

1. If p is the probability of success and q is the probability of failure, what is the probability of the sequence $SSSSFFFFFF$ in terms of p and q?
2. From result in 1), deduce the probability of a specific sequence of outcomes where there are r successes and $n-r$ failures.
3. Recall that each way of getting heads is equally likely (for example the sequence $SSSSFFFFFF$ is just as likely as the sequence $SFSFFSFFSF$). From result in 1), how many different combinations produce 4 heads?

   Considering that: The total number of ways of selecting 4 distinct combinations of 10 objects, irrespective of order, is $\dfrac{10!}{4!(10-4)!} = {}^{10}C_4$

4. From result in 3), deduce different combinations that produce r heads in n trials.

Let X be the random variable "the number of successes in the n trials".

Let p and q be the probabilities of success and failure in any one trial.

From activity 9.4, in the n independent trials, the probability that there will be r successes and $n - r$ failures is given by

$P(X=r) = {}^nC_r P^r q^{n-r}, r = 0,1,2,\ldots,n$.

The probability distribution of the random variable X is therefore given by

X	0	1	2	$\ldots r$	$\ldots n$
$P(X)$	${}^nC_0P^0q^n$	${}^nC_1P^1q^{n-1}$	${}^nC_2P^2q^{n-2}$	${}^nC_rP^rq^{n-r}$	${}^nC_nP^nq^0$

The probability distribution is called the **binomial distribution** because for r = 0, 1, 2, ..., n, $p(x)$ are the probabilities of the successive terms of the binomial expansion of $(q+p)^n$.

Binomial distribution was discovered by James-Bernoulli in 1700 and is denoted

$$b(r:n,p) = {}^nC_r P^r q^{n-r}, r = 0,1,2,\ldots,n$$

The constant n, p, q are called parameters of the binomial distribution.

Note that $p+q=1$

Notice

For N set of n trials, the successes 0, 1, 2, ...r, ..., n are given by $N(p+q)^n$, which is called binomial distribution.

Example 9.6

During war, a ship out of nine was sunk on an average in making a voyage.

What was the probability that exactly 3 out of a convoy of 6 ships would arrive safely ?

Solution

Let p be the probability of a ship arriving safely i.e.,

$$p = 1 - \frac{1}{9} = \frac{8}{9}, \text{ then } q = \frac{1}{9}, \text{ with } n = 6, N = 1$$

The binomial distribution is $N(p+q)^n = \left(\frac{8}{9} + \frac{1}{9}\right)^6$

The probability that exactly 3 ships arrive safely is

$$^6C_3\left(\frac{8}{9}\right)^3\left(\frac{1}{9}\right)^3 = 20\times\frac{512}{9^6} = \frac{10240}{9^6} = 0.0193$$

Example 9.7

The probability that a person belongs to a certain club A is 0.6. Find the probability that in a randomly selected sample of 8 people there are:

a) Exactly 3 people who belong to club A.

b) More than 5 people who belong to club A.

Solution

In this case, success is belonging to club A and failure: not belonging to club A.

$\Rightarrow p = 0.6,\ q = 1-0.6 = 0.4,$ with $n = 8, N = 1$

So, the binomial distribution is $N(p+q)^n = (0.6, 0.4)^8$

Thus,

a) $P(X=3) = {}^8C_3 p^3 q^5$

$$= \frac{8!}{3!5!}(0.6)^3(0.4)^5$$
$$= 0.124$$

b) $P(X>5) = P(X=6)+P(X=7)+P(X=8)$

$$= {}^8C_6 p^6 q^2 + {}^8C_7 p^7 q^1 + {}^8C_8 p^8 q^0$$
$$= {}^8C_6(0.6)^6(0.4)^2 + {}^8C_7(0.6)^7(0.4)^1 + {}^8C_8(0.6)^8(0.4)^0$$
$$= 0.316$$

Example 9.8

The probability that a pen drawn at random from a box of pens, is defective, is 0.1.

If a sample of 6 pens is taken, find the probability that it will contain:

a) No defective pen.

b) 5 or 6 defective pens.

c) Less than 3 defective pens.

Solution

Success: defective pen

Failure: no defective pen

$\Rightarrow p = 0.1,\ q = 1 - 0.1 = 0.9$ with $n = 6$

a) $P(X = 0) = {}^{6}C_{0}(0.1)^{0}(0.9)^{6} = 0.531$

b) $P(X > 4) = P(X = 5) + P(X = 6)$

$$= {}^{6}C_{5}(0.1)^{5}(0.9)^{1} + {}^{6}C_{6}(0.1)^{6}(0.9)^{0}$$
$$= 0.000055$$

c) $P(X < 3) = P(X = 2) + P(X = 1) + P(X = 0)$

$$= {}^{6}C_{2}(0.1)^{2}(0.9)^{4} + {}^{6}C_{1}(0.1)^{1}(0.9)^{5} + {}^{6}C_{0}(0.1)^{0}(0.9)^{6}$$
$$= 0.98$$

Example 9.9

The following data shows the number of seeds germinating out of 10 on damp filter for 80 sets of seeds. Fit a binomial distribution to this data.

x	0	1	2	3	4	5	6	7	8	9	10
f	6	20	28	12	8	6	0	0	0	0	0

Solution

Let us first find mean $\bar{x} = \dfrac{\sum x \cdot f}{\sum f}$

x	0	1	2	3	4	5	6	7	8	9	10	Total
f	6	20	28	12	8	6	0	0	0	0	0	80
$x \cdot f$	0	20	56	36	32	30	0	0	0	0	0	174

$$\bar{x} = \frac{174}{80} = \frac{87}{40}$$

The mean of a binomial distribution = np

Given that n = 10, N = 80 and $\bar{x} = \frac{87}{40}$

Therefore, $np = \frac{87}{40} \Rightarrow p = \frac{87}{400} = 0.2175$

Then, $q = 1 - p = 0.7825$

Hence, the binomial distribution to be fitted to the data is

$$N(p+q)^n = 80(0.2175 + 0.7825)^{10}$$

Thus, the expected frequencies are

x	0	1	2	3	4	5	6	7	8	9	10
f	6.9	19.1	24.0	17.8	8.6	2.9	0.7	0.1	0	0	0

Application activity 9.16

1. Find the probability of getting 4 heads in 6 tosses of fair coin.
2. If on an average one ship in every ten is wrecked during a war, find the probability that out of 5 ships expected to arrive, at least 4 will arrive safely.
3. The averall percentage of failures in a certain examination is 20. If six candidates appear in the examination, what is the probability that at least five will pass the examination?
4. Ten percent of screws produced in a certain factory turn out to be defective. Find the probability that in a sample of 10 screws chosen at random, exactly two will be defective.
5. The probability that a man aged 60 will live to be 70 is 0.65. What is the probability that out of 10 men, now 60, at least 7 will live to be 70?
6. If 10% of bolts produced by a machine are defective. Determine the probability that out of 10 bolts chosen at random;
   a) 1
   b) none
   c) utmost 2 bolts will be defective.
7. A die is thrown 8 times. Determine the probability that a 3 will be shown
   a) exactly 2 times,

b) at least seven times,

c) at least once.

8. An underground mine has 5 pumps installed for pumping out storm water. If the probability of any one of the pumps failing during the storm is $\frac{1}{8}$. What is the probability that;

a) at least 2 pumps will be working,

b) all the pumps will be working during a particular storm?

Expected value, variance and standard deviation of a binomial distribution

Activity 2.8

An experiment, or trial, whose outcome can be classified as either a success or failure is performed; $x=1$ when the outcome is the success or $x=0$ when the outcome is a failure. For any Bernoulli trial, probability of success (1) is p and probability of fail (0) is $1-p=q$.

1. Use the expected formula, $E(X)=\sum_{i=1}^{n} x_i P(X=x_i)$,

   to find the mean of any Bernoulli trial. And hence $E(X)$ for n trials.

2. Recall that, the variance for a random variable is $Var(X)=E(X^2)-[E(X)]^2$. Use this relation and result in 1) to find $Var(X)=E(X^2)-[E(X)]^2$ for n trials. Remember that $E(X^2)=\sum_{i=1}^{n} x_i^2 P(X=x_i)$

From activity 9.5,

The expected value (or mean) of a binomial distribution of a discrete random variable X is denoted and defined by

$\mu=E(X)=np$.

where n is the number of trials and p is the probability of success.

The variance of a binomial distribution of a discrete random variable X is denoted and defined by

$$\sigma^2 = Var(X) = npq = np(1-p)$$

where n is the number of trials, p is the probability of success and q is the probability of failure.

The standard deviation of a binomial distribution of a discrete random variable X is denoted and defined by $\sigma = \sqrt{Var(X)} = \sqrt{npq}$.

Example 9.10

A die is tossed thrice. If getting an even number is considered as success, what is the variance of the binomial distribution?

Solution

Let p be the probability of getting an even number,

i.e. $p = \frac{3}{6} = \frac{1}{2}$, then $q = 1 - \frac{1}{2} = \frac{1}{2}, n = 3$

The variance of binomial distribution $= npq = 3 \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$

Example 9.11

The mean and variance of a binomial distribution are 4 and 3 respectively.

Find the probability of getting exactly six successes in this distribution.

Solution

The mean of binomial distribution $= np = 4$ (1)

And the variance of binomial distribution $= npq = 3$ (2)

Using (1) and (2), we have

$$npq = 3 \Rightarrow 4 \cdot q = 3 \Rightarrow q = \frac{3}{4}$$

and $p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$

From (1), $np = 4 \Rightarrow n \times \frac{1}{4} = 4 \Rightarrow n = 16$

The probability of 6 successes; ${}^{16}C_6\left(\frac{1}{4}\right)^6\left(\frac{3}{4}\right)^{10} = \frac{8008\times 3^{10}}{4^{16}}$

Example 9.12

In 256 sets of 12 tosses of a coin, how many cases one can expect 8 heads and 4 tails.

Solution

Let p be the probability of head and q be the probability of tail, thus

$$p=\frac{1}{2}\ and\ \ q=\frac{1}{2}; n=12, N=256$$

The binomial distribution is

$$N(p+q)^n = 256\left(\frac{1}{2}+\frac{1}{2}\right)^{12}$$

The probability of 8 heads and 4 tails in 12 trials is

$$ {}^{12}C_8\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^4 = \frac{495}{4096}$$

The expected number of such cases in 256 sets $256\times\frac{495}{4096} = 30.9 \simeq 31($

Example 9.13

A random sample is taken on 800 families with 4 children each, how many families would be expected to have;

(i) 2 boys and 2 girls (ii) at least one boy

(iii) no girl (iv) at most two girls?

Assuming equal probabilities for boys and girls.

Solution

Since, the probabilities for boys and girls are equal, let p be the probability of having a boy and q be the probability of having a girl.

Here, $p=q=\frac{1}{2}$, n = 4, N = 800

The binomial distribution is $N(p+q)^n = 800\left(\frac{1}{2},\frac{1}{2}\right)^4$.

(i) The expected number of families having 2 boys and 2 girls is

$$800\,{}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{800\times 6}{16} = 300$$

ii) The expected number of families having at least one boy is

$$800\left({}^4C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 + {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + {}^4C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^1 + {}^4C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^0\right)$$

$$= \frac{800\times(4+6+4+1)}{16} = 750$$

(iii) The expected number of families having no girl is

$$800\,{}^4C_4\left(\frac{1}{2}\right)^4 = \frac{800}{16} = 50$$

(iv) The expected number of families having utmost two girls is

$$800\left({}^4C_0\left(\frac{1}{2}\right)^4 + {}^4C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^3 + {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2\right)$$

$$= \frac{800\times(1+4+6)}{16} = 550$$

Application activity 9.17

1. Bring out the fallacy in the statement:
   "The mean of a binomial distribution is 3 and variance is 4".
2. A die is tossed 180 times. Find the mean and the standard deviation of the random variable representing the total number of sixes obtained.
3. A card is selected from an ordinary deck of 52 cards then replaced before a second card is selected. This procedure is carried out 10 times. If X represents the number of spades selected, find;

   a) $E(X)$ b) $\text{Var}(X)$

4. If in a binomial experiment of n trials, the probability of succes is p and the mean and variance are 3 and 2 respectively, find the probability that there is:

   a) Exactly 1 success.

   b) At least 1 success.

5. An irregular six-faced die is thrown and the expectation that in 100 throws, it will give five even numbers is twice the expectation that it will give four even numbers. How many times in 10,000 sets of 10 throws would you expect it to give no even number?

6. In a precision bombing attack, there is a 50% chance that anyone bomb will strike the target. Two direct hits are required to destroy the target completely. How many bombs must be dropped to give a 99% chance or better of completely destroying the target?

7. Assuming that half the population are consumers of rice so that the chance of an individual being a

   consumer is $\frac{1}{2}$ and assuming that 100 investigators,

each take ten individuals to see whether they are consumers or not, how many investigators do you expect to report that three people or less are consumers ?

9.1.4. Uncountable infinite discrete case: Poisson distribution

Definition

Activity 2.9

Consider the Maclaurin's expansion of e^{θ}. That is,

$$e^{\theta}=\frac{\theta^0}{0!}+\frac{\theta^1}{1!}+\frac{\theta^2}{2!}+\frac{\theta^3}{3!}+\ldots+\frac{\theta^n}{n!}+\ldots \quad (1)$$

In a probability situation, the sums of the probabilities for all the possible outcomes must sum to 1, then, we can say that, any algebraic sum whose value is 1 can in theory be interpreted as a probability distribution.

1. Divide both sides of relation (1) by e^{θ} to get 1 on the left hand side.
2. Suppose that θ in relation obtained in 1) is the rate of occurrence of a random event per some unit or module of observation say; an hour of time, or a yard of length, or whatever. Substitute λ (rate) for the previous θ to obtain new relation.
3. Suppose that we want to know, given a particular average rate of occurrence (λ), how many times x events will most likely be observed in a set of n time or space units. Use the general term of the series obtained in 2) and put $n = x$ to get the probability $P(X = x)$.

Poisson distribution was discovered by a French mathematician Simeon Denis Poisson in 1837. Poisson distribution is also a discrete probability distribution of a discrete random variable, which has no upper bound.

The **Poisson distribution** is a discrete distribution often used as a model for the number of events (such as the number of customers in waiting lines, number of defects in a given surface area, airplane arrivals, or the number of accidents at an intersection) in a specific time period.

Poisson distribution is a limiting form of the binomial distribution $(p+q)^n$ under the following conditions:

a) $n \to \infty$, i.e., the number of trials is indefinitely large.

b) $p \to 0$, i.e., the constant probability of success for each trial is indefinitely small.

c) np is a finite quantity, say λ.

Thus, $p = \frac{\lambda}{n}$, $q = 1 - \frac{\lambda}{n}$, where λ is a positive real number.

The major difference between Poisson and Binomial distributions is that the Poisson distribution does not have a fixed number of trials but it instead uses a fixed number of time or space in which the number of success is recorded.

Typical events which could have a Poisson distribution are:

a) Number of customers arriving at a supermarket checkout per minute.

b) Number of suicides or deaths caused by heart attack in 1 minute.

c) Number of accidents that take place on a busy road in time *t*.

d) Number of printing mistakes at each unit of the book.

e) Number of cars passing a certain street in time *t*.

f) Number of $\alpha-$ particles emmited per second by radioactive sources.

g) Number of faulty blades in a packet of 1000.

h) Number of persons born blind per year in a certain village.

i) Number of telephone calls received at a particular switch board in a minute.

j) Number of times a teacher is late for class in a given week.

From activity 9.6, the probability density function of Poisson distribution is defined by

$$P(X=x)=\frac{e^{-\lambda}\lambda^{x}}{x!},\quad x=0,1,2,\ldots$$

Where λ is a parameter which indicates the average number (the expected value) of events in the given time interval and $e\approx 2.718\ldots$

For a Poisson distribution, we write $X\sim \mathrm{Po}(\lambda)$

Notice

- If the initial probability is known, you can find successive probabilities using the following recurrence relation;

$$P(X=x+1)=\frac{\lambda}{x+1}P(X=x).$$

Indeed, $P(X=x)=\dfrac{e^{-\lambda}\lambda^{x}}{x!}\quad (1)$ and

$$P(X=x+1)=\frac{e^{-\lambda}\lambda^{(x+1)}}{(x+1)!}\quad (2)$$

Dividing (2) by (1), we get

$$\frac{P(X=x+1)}{P(X=x)}=\frac{\dfrac{e^{-\lambda}\lambda^{x+1}}{(x+1)!}}{\dfrac{e^{-\lambda}\lambda^{x}}{x!}}=\frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!}\frac{x!}{e^{-\lambda}\lambda^{x}}$$

$$=\frac{e^{-\lambda}\lambda^{x}\lambda}{(x+1)x!}\frac{x!}{e^{-\lambda}\lambda^{x}}=\frac{\lambda}{x+1}$$

Thus,

$$P(X=x+1)=\frac{\lambda}{x+1}P(X=x)$$

- For a Poisson distribution of a discrete random variable X, the mean μ (or expected value) and the variance σ^2 are the same and equal to λ. Thus, $\mu=\sigma^2=\lambda$.

Example 9.14

On average on Friday, a waitress gets no tip from 5 customers. Find the probability that she will get no tip from 7 customers this Friday.

Solution

The waitress averages 5 customers that leave no tip on Friday: $\Rightarrow \lambda=5$.

But $x=7$

$$\Rightarrow P(X=7)=\frac{e^{-5}\times 5^7}{7!}=0.104$$

Example 9.15

A small life insurance company has determined that on the average, it receives 6 death claims per day. Find the probability that the company receives at least seven death claims on a randomly selected day.

Solution

Since $\lambda=6$, and we need $P(X\geq 7)$

$$P(X\geq 7)=1-P(X\leq 6)$$

$$\begin{aligned}P(X\leq 6)&=P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6)\\&=\frac{e^{-6}\times 6^0}{0!}+\frac{e^{-6}\times 6^1}{1!}+\frac{e^{-6}\times 6^2}{2!}+\frac{e^{-6}\times 6^3}{3!}+\frac{e^{-6}\times 6^4}{4!}+\frac{e^{-6}\times 6^5}{5!}+\frac{e^{-6}\times 6^6}{6!}\\&=0.6265\end{aligned}$$

$$P(X\geq 7)=1-0.6265=0.3735$$

Example 9.16

The number of traffic accidents that occur on a particular stretch of road during a month follows a Poisson distribution with a mean of 9.4. Find the probability that less than two accidents will occur on this stretch of road during a randomly selected month.

Solution

Here $\lambda = 9.4$, we need $P(X < 2)$

$$P(X<2) = P(X=0) + P(X=1)$$

$$= \frac{e^{-9.4}\times(9.4)^0}{0!} + \frac{e^{-9.4}\times(9.4)^1}{1!} = 0.00085$$

Example 9.17

If the variance of the Poisson distribution is 2, find the probabilities for x = 1, 2, 3, 4 from the recurrence relation of the Poisson distribution. Also, obtain $P(x \geq 4)$.

Solution

We know that for Poisson's distribution, mean and variance are both equal i.e., mean = variance = 2.

Recurrence relation for Poisson distribution

$$P(x+1) = \frac{\lambda}{x+1}P(x) \Rightarrow P(x+1) = \frac{2}{x+1}P(x)$$

The Poisson's distribution is

$$\Rightarrow P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

From recurrence relation for Poisson distribution,

if $x=0, P(1) = \frac{2}{0+1}P(0) = 2\times 0.1353 = 0.2706$

if $x=1, P(2) = \frac{2}{1+1}P(1) = 0.2706$

if $x=2, P(3)=\frac{2}{2+1}P(2)=\frac{2\times 0.2706}{3}=0.1804$

if $x=3, P(4)=\frac{2}{3+1}P(3)=\frac{1}{2}\times 0.1804=0.0902$

Now, $P(x\geq 4)=P(4)+P(5)+P(6)+\ldots$

$=1-\left[P(0)+P(1)+P(2)+P(3)\right]$

= 1 – [0.1353 + 0.2706 + 0.2706 + 0.1804]

= 1 – 0.8569 = 0.1431.

Application activity 9.18

1. Criticise the following statement:
   "The mean of a Poisson distribution is 7, while the standard deviation is 6".
2. The number of road construction projects that take place at any one time in a certain city follows a Poisson distribution with a mean of 7. Find the probability that more than four road construction projects are currently taking place in the city.
3. The number of traffic accidents that occur on a particular stretch of road during a month follows a Poisson distribution with a mean of 7.

   Find the probability of observing exactly three accidents on this stretch of road in the next month.
4. Suppose the number of babies born during an 8-hour shift at a hospital's maternity wing follows a Poisson distribution with a mean of 6 an hour, find the probability that five babies are born during a particular 1-hour period in this maternity wing.
5. The university policy department must write, on average, five tickets per day to keep department revenues at budgeted levels. Suppose the number of tickets written per day follows a Poisson distribution with a mean of 8.8 tickets per day. Find the probability that less than six tickets are written on a randomly selected day from this distribution.
6. The number of goals scored at State College hockey games follows a Poisson distribution with a mean of 3 goals per game. Find the probability that each of four randomly selected State College hockey games resulted in six goals being scored.

7. Red blood cell deficiency may be determined by examining a specimen of the blood under a microscope. Suppose a certain small fixed volume contains on the average 20 red cells for normal persons. Using Poisson distribution, obtain the probability that a specimen from a normal person will contain less than 15 red cells.

8. A skilled typist, on routine work, kept a record of mistakes made per day during 300 working days

Number of cells per square (x)	0	1	2	3	4	5	6
Number of squares (f)	103	143	98	42	8	4	2

Fit a Poisson distribution to the above data and hence calculate the expected (theoretical) frequencies.

9.2. Continuous random variables

9.2.1. Probability density function

Activity 9.10

Given the function $f(x)=\begin{cases} k(x+1)^2 & -1\leq x\leq 0 \\ k & 0<x\leq 1 \end{cases}$;

1. Find the value of the constant k if the area under the curve of $f(x)$ is 1.

The Area under a curve is determined by integration.

2. Sketch the graph of $f(x)$.

A random variable X is said to be continuous if its possible values are all real values in some interval. A **continuous random variable** is theoretical representation of continuous variable such as weight, temperature, time, distance, mass and height.

To describe the probability of a continuous random variable, we use a **probability density function** (p.d.f.) $f(x)\geq 0$.

A function defined on an interval $[a,b]$ is a probability density function for a continuous random variable X distributed on $[a,b]$ if, whenever x_1 and x_2 satisfy $a \le x_1 \le x_2 \le b$, we have

$$p(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(x)\,dx.$$

If X is a continuous random variable, then the probability that the values of X will fall between the values a and b is given by the area of the region lying below the graph of $f(x)$ and above the $x-axis$ between a and b and this area is $\int_a^b f(x)\,dx$.

We have $p(a \le x \le b) = \int_a^b f(x)\,dx$ and $f(x) \ge 0$ for $a \le x \le b$.

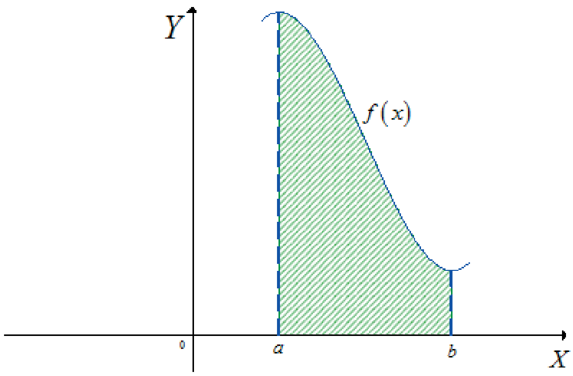


Figure 9.1. *Area enclosed by the curve of function f(x) and x-axis*

Properties of pdf, $f(x)$

a) $f(x) > 0$ for all x

b) $\int_{all\,x} f(x)\,dx = 1$

How to obtain probabilities

The probability that a random variable attains values between x_1 and x_2 given by $P(x_1 \le x \le x_2)$ is obtained from the area under the curve between x_1 and x_2.

Therefore, $P(x_1 \le x \le x_2) = \int_{x_1}^{x_2} f(x)\,dx$.

Example 9.18

A continuous random variable x has a p.d.f

$$f(x)=\begin{cases} kx, 0\le x\le 4 \\ 0, otherwise \end{cases}$$

where k is a constant.

a) Find the value of the constant k.

b) Sketch the graph $y=f(x)$.

c) Find $P(1\le x\le 2.5)$.

Solution

a) $\int_a^b f(x)=1$

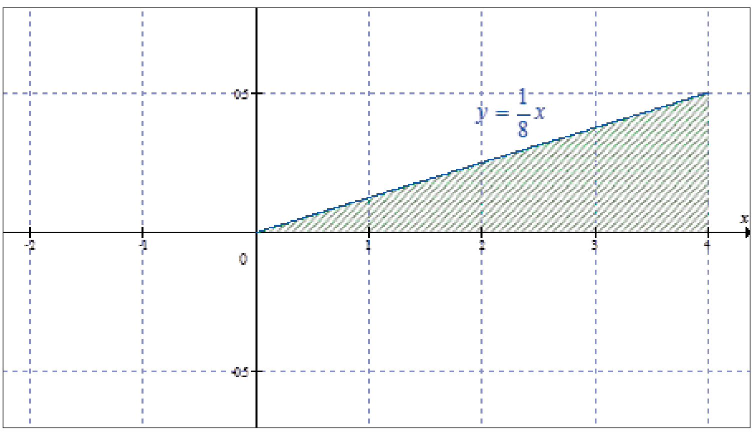
$$\int_0^4 kx=1\Rightarrow\left[\frac{kx^2}{2}\right]_0^4=1\Rightarrow\frac{16k}{2}=1\Rightarrow k=\frac{1}{8}$$

b) Graph

$$y=f(x)\Rightarrow y=\frac{1}{8}x$$

If $x=0,\ y=0$

If $x=4,\ y=\frac{1}{2}$



c) $P(1 \le x \le 2.5) = \int_1^{2.5} \frac{1}{8} x dx$

$$= \left[\frac{1}{16}x^2\right]_1^{2.5} = \frac{1}{16}\left[(2.5)^2 - 1\right] = \frac{21}{64}$$

Example 9.19

A continuous random variable x has a p.d.f $f(x)$ where

$$f(x) = \begin{cases} kx, & 0 \le x < 2, \\ k(4-x), & 2 \le x \le 4, \\ 0, & \text{else where;} \end{cases}$$

where k is a constant.

a) Find the value of the constant k.

b) Sketch $y = f(x)$

c) Find $P\left(\frac{1}{2} \le x \le 2.5\right)$

Solution

a) $\int_0^2 kxdx + \int_2^4 k(4-x)dx = 1$

$$\Rightarrow \left[\frac{kx^2}{2}\right]_0^2 + \left[4kx - \frac{kx^2}{2}\right]_2^4 = 1$$

$$\Rightarrow 2k + 16k - 8k - 8k + 2k = 1 \Rightarrow 4k = 1 \Rightarrow k = \frac{1}{4}$$

$$f(x) = \begin{cases} kx, & 0 \le x < 2 \\ k(4-x), & 2 \le x \le 4 \\ 0, & \text{else where} \end{cases} \Rightarrow f(x) = \begin{cases} \frac{1}{4}x, & 0 \le x < 2 \\ \frac{1}{4}(4-x), & 2 \le x \le 4 \\ 0, & \text{else where} \end{cases}$$

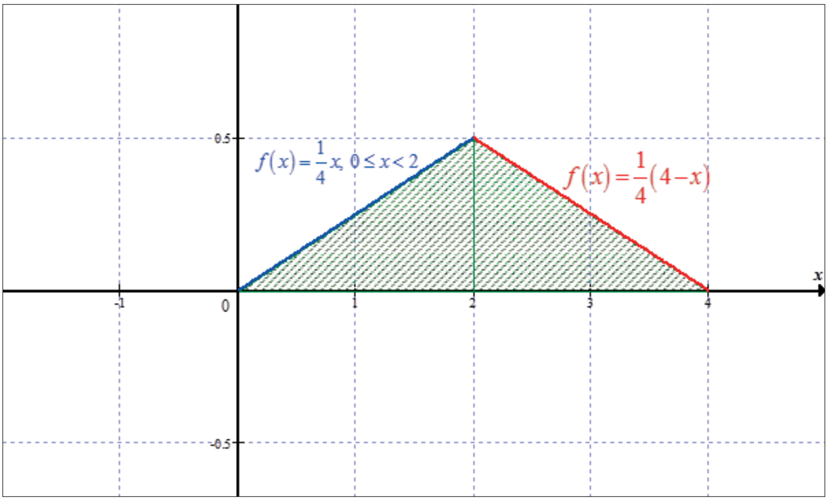
b) Sketch of $f(x)$

$$f(x) = \frac{1}{4}x, \quad 0 \le x < 2,$$

If $x=0,\ y=0$, if $x=2,\ y=\dfrac{1}{2}$

$f(x)=\dfrac{1}{4}(4-x),\quad 2\le x\le 4$,

If $x=2,\ y=\dfrac{1}{2}$, if $x=4,\ y=0$



c) $P\left(\dfrac{1}{2}\le x\le 2.5\right)=P\left(\dfrac{1}{2}\le x\le 2\right)+P(2\le x\le 2.5)$

$$=\int_{0.5}^{2}\frac{1}{4}x\,dx+\int_{2}^{2.5}\frac{1}{4}(4-x)\,dx$$

$$=\left[\frac{x^2}{8}\right]_{0.5}^{2}+\left[\frac{1}{4}\left(4x-\frac{x^2}{2}\right)\right]_{2}^{2.5}$$

$$=\frac{4}{8}-\frac{(0.5)^2}{8}+\frac{1}{4}\left[4(2.5)-\frac{(2.5)^2}{2}-8+\frac{4}{2}\right]$$

$$=\frac{11}{16}$$

Application activity 9.19

1. Let X be a random variable with a p.d.f given by

$$f(x)=\begin{cases} cx^2, & |x|\le 1, \\ 0, & \text{otherwise}; \end{cases}$$

where c is a constant.

a) Find the constant c

b) Find $P\left(X\ge \frac{1}{2}\right)$

2. Triangle ABC is right angled at B and $AC=10\,cm$. If $BC=X\,cm$ and X is a random variable uniformly distributed between $6\,cm$ and $8\,cm$, find the probability that the length of AB exceeds $7.5\,cm$.

3. A continuous random variable X distributed has p.d.f of the form

$$f(x)=\begin{cases} \text{k}, & 0\le x\le 2, \\ k(2x-3), & 2\le x\le 3, \\ 0, & otherwise; \end{cases}$$

where c is a constant.

Determine;

a) the value of constant k.

b) (i) $P(x<1)$ ii) $P(x=1)$

iii) $P(x>2.5)$ iv) $P[(0<x<2)/x\ge 1]$

Hint: $P(A/B)=\dfrac{P(A\cap B)}{P(B)}$

Cumulative distribution of continuous random variable

Activity 2.11

Consider a continuous random variable X with probability density function: $f(x)=\begin{cases}\frac{1}{4}x, & 1\le x\le 3\\ 0, & \text{otherwise}\end{cases}$

Let $F(x)=\int_1^x f(x)\,dx$ be the cumulative distribution function.

1. Find $F(x)$ for $x<1$
2. Find $F(x)$ for $1\le x\le 3$
3. Find $F(x)$ for $x>3$
4. Combine results from 1) to 3) and write down the function $F(x)$.

Hint: Recall that $F(x)$ accumulates all of the probability less than or equal to x.

You might recall that the cumulative distribution function is defined for discrete random variables. Again, $F(x)$ accumulates all of the probability less than or equal to x. The cumulative distribution function for continuous random variables is just a straight forward extension of that of the discrete case. All we need to do is replace the summation with an integral.

The **cumulative distribution** function of a continuous random variable X is defined as $F(x)=\int_{-\infty}^{x} f(t)\,dt$ for $x\in\,]-\infty,+\infty[$.

Properties

- $F(x)=0\ for\ x\to-\infty$
- $F(x)=1$ for $x\to+\infty$

Example 9.20

A continuous random variable X has the following probability density function: $f(x)=\begin{cases}3x^2, 0<x<1\\ 0, elsewhere\end{cases}$.

What is the cumulative distribution function $F(x)$?

Solution

For $x \leq 0$, $f(x)=0$ and $F(x)=0 \Rightarrow F(0)=0$

$$F(x)=\int_0^x 3t^2\,dt$$
$$=\left[t^3\right]_0^x$$
$$=x^3, \quad 0<x<1$$

For $x \geq 1$, $f(x)=0$ and $F(x)=F(1)+\int_1^x f(t)\,dt=1+0=1$

Then,

$$F(x)=\begin{cases} 0, & x \leq 0, \\ x^3, & 0<x<1, \\ 1, & x \geq 1. \end{cases}$$

Example 9.21

A continuous random variable X has the following probability density function: $f(x)=\begin{cases} \dfrac{x^3}{4}, 0<x<2 \\ 0, otherwise \end{cases}$.

What is the cumulative distribution function $F(x)$?

Solution

For $x \leq 0$, $f(x)=0$ and $F(x)=0 \Rightarrow F(0)=0$.

$$F(x)=\int_0^x \frac{t^3}{4}\,dt$$
$$=\left[\frac{t^4}{16}\right]_0^x$$
$$=\frac{x^4}{16}, \quad 0<x<2$$

For $x \geq 2$, $f(x)=0$ and $F(x)=F(1)+\int_2^x f(t)\,dt=1+0=1$.

Then,

$$F(x)=\begin{cases}0, & x\le 0\\ \dfrac{x^4}{16}, & 0<x<2\\ 1, & x\ge 2\end{cases}$$

Application activity 9.20

1. Suppose that the probability density function of a continuous random variable X is defined as

$$f(x)=\begin{cases}x+1, & -1<x<0\\ 1-x, & 0\le x<1\end{cases}$$

Find the cumulative density function.

2. Given the probability density function of a continuous random variable X defined as

$$f(x)=\begin{cases}\dfrac{x}{8}, & 0\le x\le 4\\ 0, & \text{otherwise}\end{cases}$$

Find the cumulative density function.

3. A probability density function of a continuous random variable X is defined by

$$f(x)=\begin{cases}\dfrac{x}{3}, & 0\le x\le 2\\ \dfrac{-2x}{3}+2, & 2<x\le 3\\ 0, & \text{otherwise}\end{cases}$$

Find the cumulative density function.

9.2.2. Expected value, variance and standard deviation

Activity 9.12

A continuous random variable X takes values between 0 and 1 and has a probability density function $f(x)=6x(1-x)$. Find:

1. $A=\int_0^1 xf(x)dx$ 2. $B=\int_0^1 x^2 f(x)dx$ 3. $B-A^2$

If X is a continuous random variable with probability density function $f(x)$ on interval $[a,b]$, then,

the **mean** μ (or **expected value** $E(X)$) of X is denoted and defined by

$$\mu=E(X)=\int_a^b xf(x)dx.$$

Also, expectation of function *g(X)* is

$$E(g(x))=\int_a^b g(x)f(X)dx.$$

The **variance** $Var(x)$ or σ^2 is denoted and defined by

$$\sigma^2=Var(X)=E(X^2)-[E(x)]^2.$$

And the standard deviation is

$$\sigma=SD=\sqrt{Var(X)}.$$

Notice

In $\mu=E(X)=\int_a^b xf(x)dx$, $[a,b]$ is the interval where $f(x)$ is defined.

Generally, $\mu=E(X)=\int_{-\infty}^{+\infty} xf(x)dx$ and

$$\sigma^2=Var(X)=\int_{-\infty}^{+\infty} x^2 f(x)dx-[E(x)]^2.$$

Properties of $E(X)$ and $Var(X)$

$\forall a, b \in \mathbb{R}$

1. $E(a) = a$
2. $E(aX) = aE(X)$
3. $E(aX + b) = aE(X) + b$
4. $E(X + Y) = E(X) + E(Y)$
5. $E(aX^2 + b) \neq aE(X^2) + b$
6. $\text{var}(a) = 0$
7. $\text{var}(aX) = a^2 \text{var}(X)$
8. $\text{var}(aX + b) = a^2 \text{var}(X)$

Example 9.22

If X is a continuous random variable with probability density function

$$f(x) = \begin{cases} \frac{3x^2}{64}, 0 \leq x \leq 4 \\ 0, \text{elsewhere} \end{cases}$$

Find the expected value $E(X)$.

Solution

$$E(X) = \int_a^b x f(x) dx$$

$$E(X) = \int_0^4 x\left(\frac{3x^2}{64}\right) dx \quad = \int_0^4 \frac{3x^3}{64} dx \quad = \left[\frac{3x^4}{256}\right]_0^4 \quad = \frac{3(4)^4}{256} - 0 = 3$$

Example 9.23

The continuous random variable has a probability density function

$$f(x) = \begin{cases} \frac{1}{20}(x+3), 0 \leq x \leq 4 \\ 0, otherwise \end{cases}$$

a) Find $E(X)$

b) Verify that $E(2X + 5) = 2E(X) + 5$

c) Find $E(X^2)$

d) Find $E\left(X^2+2X+3\right)$

Solution

a) $E(X)=\int_0^4 x\frac{(x+3)}{20}dx$

$$=\int_0^4 \frac{1}{20}\left(x^2+3x\right)dx \quad =\left[\frac{1}{20}\left(\frac{x^3}{3}+\frac{3x^2}{2}\right)\right]_0^4$$

$$=\frac{1}{20}\left(\frac{4^3}{3}+\frac{3\times 4^2}{2}-0\right)$$

$$=\frac{34}{15}$$

b) $E(2X+5)=\int_0^4(2x+5)\frac{1}{20}(x+3)dx \quad =\int_0^4\frac{1}{20}\left(2x^2+11x+15\right)dx$

$$=\left[\frac{1}{20}\left(\frac{2x^3}{3}+\frac{11x^2}{2}+15x\right)\right]_0^4$$

$$=\frac{1}{20}\left(\frac{2(4)^3}{3}+\frac{11(4)^2}{2}+15(4)-0\right)$$

$$=\frac{143}{15}$$

On the other hand,

$$2E(X)+5=2\left(\frac{34}{15}\right)+5 \quad \textit{from result in a)}$$

$$=\frac{143}{15}$$

Thus, $E(2X+5)=2E(X)+5=\frac{143}{15}$ hence verified

c) $E\left(X^2\right)=\int_0^4 x^2\frac{(x+3)}{20}dx$

$$=\int_0^4 \frac{1}{20}\left(x^3+3x^2\right)dx \quad =\left[\frac{1}{20}\left(\frac{x^4}{4}+\frac{3x^3}{3}\right)\right]_0^4$$

$$=\frac{1}{20}\left(\frac{4^4}{3}+4^3-0\right)$$

$$=\frac{32}{5}$$

d) $E\left(X^2+2X+3\right)=E\left(X^2\right)+2E\left(X\right)+3 \quad \textit{from properties}$

$$=\frac{32}{5}+2\frac{34}{15}+3 \qquad \textit{from results in } a)\textit{ and } c)$$

$$=\frac{209}{15}$$

Example 9.24

The continuous random variable has a probability density function

$$f(x)=\begin{cases}\frac{x}{8}, 0\le x\le 4\\ 0, elsewhere\end{cases}$$

Find:

a) $E(X)$ b) $E\left(X^2\right)$ c) $\text{var}(X)$

d) $\text{SD}(X)$ e) $\text{var}(3X+2)$

Solution

a) $E(X)=\int_0^4 x\frac{x}{8}dx \quad =\int_0^4 \frac{x^2}{8}dx \quad =\left[\frac{x^3}{24}\right]_0^4$

$$=\frac{8}{3}$$

b) $E\left(X^2\right)=\int_0^4 x^2\frac{x}{8}dx \quad =\int_0^4 \frac{x^3}{8}dx \quad =\left[\frac{x^4}{32}\right]_0^4$

$$=8$$

c) $\operatorname{var}(X) = E(X^2) - [E(X)]^2 = 8 - \left(\frac{8}{3}\right)^2 = \frac{8}{9}$

d) $\operatorname{SD}(X) = \sqrt{\operatorname{var}(X)} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$

e) $\operatorname{var}(3X+2) = 3^2 \operatorname{var}(X) = 9 \times \frac{8}{9} = 8$

Example 9.25

Find the mean μ and the standard deviation σ of a random variable X distributed uniformly on the interval $[a,b]$. Find $P(\mu - \sigma \le x \le \mu + \sigma)$.

Solution

The probability density on the interval is $f(x) = \frac{1}{b-a}$ on $[a,b]$, so the mean is given by

$$\mu = E(X) = \int_a^b \frac{x}{b-a}\,dx = \left(\frac{1}{b-a}\right) \frac{x^2}{2}\Bigg|_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2}$$

$$\sigma^2 = E(X^2) - [E(X)]^2$$

But $E(X^2) = \int_a^b \frac{x^2}{b-a}\,dx = \frac{1}{b-a}\frac{x^3}{3}\Bigg|_a^b = \frac{b^3 - a^3}{3(b-a)} = \frac{b^2 + ba + a^2}{3}$

Hence the variance is

$$\sigma^2 = \frac{b^2 + ba + a^2}{3} - \left(\frac{b+a}{2}\right)^2 = \frac{b^2 + ba + a^2}{3} - \frac{b^2 + 2ba + a^2}{4}$$

$$= \frac{4b^2 + 4ba + 4a^2 - 3b^2 - 6ba - 3a^2}{12} = \frac{b^2 - 2ba + a^2}{12} = \frac{(b-a)^2}{12}$$

Therefore, standard deviation is $\sigma = \frac{b-a}{2\sqrt{3}}$

Finally,

$$P(\mu - \sigma \le x \le \mu + \sigma) = \int_{\mu-\sigma}^{\mu+\sigma} \frac{dx}{b-a} = \left(\frac{1}{b-a}\right) x \Bigg|_{\mu-\sigma}^{\mu+\sigma}$$

$$=\frac{1}{b-a}\left(\mu+\sigma-(\mu+\sigma)\right)$$

$$=\frac{2\sigma}{b-a}=\frac{b-a}{(b-a)\sqrt{3}}=\frac{1}{\sqrt{3}}\approx 0.577$$

Application activity 9.21

1. A continuous random variable X has a probability density function defined by $f(x)=\begin{cases}\frac{k}{1+x}, & 0\le x\le 1,\\ 0, & \text{elsewhere}.\end{cases}$

   where k is a constant.

   Find the; a) value of k.

   b) mean and variance.

2. Let X be a random variable with a p.d.f given by

   $$f(x)=\begin{cases}cx^2, & |x|\le 1\\ 0, & otherwise\end{cases}$$

   where c is a constant.

   Find; a) the constant c

   b) $E(X)$ and $\text{var}(X)$

3. The outputs of 9 machines in a factory are independent variables each with p.d.f given by

   $$f(x)=\begin{cases}ax, & 0\le x\le 10\\ a(20-x), & 10\le x\le 20\\ 0, & elsewhere\end{cases}$$

   Find;

   a) the value of a

   b) the expected value and variance of outputs of each machine.

   Hence or otherwise, the expected value and variance of the total outputs from all machines.

Unit summary

1. Discrete and finite random variables

a) Probability density function

A random variable X is said to be a **discrete random variable**, if it takes only finite values

between its limits; for example, the number of student appearing in a festival consisting of 400 students is a discrete random variable which can assume values other than 0, 1, 2, …,400.

The **probability density function** (p.d.f), $F(x)$, is a function that allocates probabilities

to all distinct values that X can take on.

If the initial probability is known, you can find successive probabilities using the following recurrence relation

$$P(X = x+1) = \left(\frac{n-x}{x+1}\right)\left(\frac{p}{q}\right)P(X = x) \; .$$

To find a **cumulative probability,** we add the probabilities for all values qualifying as "less than or equal" to the specified value. Then, the cumulative distribution function of a random variable X is the function $F(x) = P(X \le x)$.

2. Expectation, variance and standard deviation

The **expected** value of random variable X, which is the **mean** of the probability distribution of X, is denoted and defined by

$$\mu = E(X) = \int_{-\infty}^{+\infty} xf(x)\,dx$$

Also, the expectation of any function $g(X)$ of the random variable X is

$$E(g(x)) = \int_{-\infty}^{+\infty} g(x)\,f(x)\,dx$$

The variance of random variable X is denoted and defined by

$$\sigma^2 = Var(X) = \sum_{i=1}^{n}[x_i - \mu]^2\,P(X = x_i)$$

This can be simplified to

$$\sigma^2 = Var(X) = E(X^2) - \mu^2$$

The standard deviation of random variable X, denoted by $\mathrm{SD}(X)$, is the square root of the variance. That is,

$$\sigma = SD(X) = \sqrt{Var(X)}$$

Properties for mean and variance

$\forall a, b \in \mathbb{R}$

a) $E(a) = a$

b) $E(aX) = aE(X)$

c) $E(aX+b) = aE(X)+b$

d) $E(X+Y) = E(X)+E(Y)$

e) $E(aX^2+b) \neq aE(X^2)+b$

f) $\operatorname{var}(a) = 0$

g) $\operatorname{var}(aX) = a^2 \operatorname{var}(X)$

h) $\operatorname{var}(aX+b) = a^2 \operatorname{var}(X)$

3. Binomial distribution (Law of Bernoulli)

For binomial probability distribution, we are interested in the probabilities of obtaining r successes in n trials, in other words r successes and n-r failures in n attempts.

Binomial distribution is denoted

$$b(r:n,p) = {}^nC_r P^r q^{n-r}, r = 0,1,2,\ldots,n$$

The constant n, p, q are called parameters of the binomial distribution.

The following are assumptions made:

- There is a fixed number (n) of trials.
- The probability of success is the same for each trial.
- Each trial is independent of all other trials.

Note that $p+q=1$

For N set of n trials, the successes 0, 1, 2, ...r, ..., n are given by $N(p+q)^n$, which is called binomial distribution.

The expected value (or mean) of a binomial distribution of a discrete random variable X is denoted and defined by $\mu = E(X) = np$ where n is the number of trials and p is the probability of success.

The variance of a binomial distribution of a discrete random variable X is denoted and defined by $\sigma^2 = Var(X) = npq$.

where n is the number of trials, p is the probability of success and q is the probability of failure.

The standard deviation of a binomial distribution of a discrete random variable X is denoted and defined by $\sigma = \sqrt{Var(X)} = \sqrt{npq}$.

4. Uncountable infinite discrete case: Poisson distribution

The **Poisson distribution** is a discrete distribution often used as a model for the number of events (such as the number of customers in waiting lines, number of defects in a given surface area, airplane arrivals, or the number of accidents at an intersection) in a specific time period.

Poisson distribution is a limiting form of the binomial distribution $(p+q)^n$ under the following conditions:

(i) $n \to \infty$, i.e., the number of trials is indefinitely large.

(ii) $p \to 0$, i.e., the constant probability of success for each trial is indefinitely small.

(iii) np is a finite quantity, say λ.

Typical events which could have a Poisson distribution are:

(i) Number of customers arriving at a supermarket checkout per minute.

(ii) Number of suicides or deaths by heart attack in 1 minute.

(iii) Number of accidents that take place on a busy road in time t.

(iv) Number of printing mistakes at each unit of the book.

(v) Number of cars passing a certain street in time t.

(vi) Number of $\alpha-$ particles emitted per second by a radioactive source.

(vii) Number of faulty blades in a packet of 1000.

(viii) Number of persons born blind per year in a certain village.

(ix) Number of telephone calls received at a particular switch board in a minute.

(x) Number of times a teacher is late for class in a given week.

The probability density function of P**oisson distribution** is defined by

$$P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!},\ x=0,1,2,...$$

where λ is a parameter which indicates the average number (the expected value) of events in the given time interval. We write $X \sim \text{Po}(\lambda)$.

» If the initial probability is known, you can find successive probabilities using the following recurrence relation;

$$P(X=x+1)=\frac{\lambda}{x+1}P(X=x)$$

» For a Poisson distribution of a discrete random variable X, the mean μ (or expected value) and the variance σ^2 are the same and equal to λ. Thus, $\mu=\sigma^2=\lambda$.

5. Continuous random variables

a) Probability density function

A function defined on an interval $[a,b]$ is a probability density function for a continuous random variable X distributed on $[a,b]$ if, whenever x_1 and x_2 satisfy $a \le x_1 \le x_2 \le b$, we have $p(x_1 \le X \le x_2)=\int_{x_1}^{x_2} f(x)dx$

Properties of p.d.f $f(x)$

(i) $f(x)>0$ for all x

(ii) $\int_{all\,x} f(x)dx=1$

The **cumulative distribution** function of a continuous random variable X is defined as: $F(x)=\int_{-\infty}^{x} f(t)\,dt$

where

- $F(x)=0$ for $x \to -\infty$
- $F(x)=1$ for $x \to +\infty$

b) Expected value, variance and standard deviation

The **mean** μ (or **expected value** $E(X)$) of X is denoted and defined by

$$\mu=E(X)=\int_{-\infty}^{+\infty} xf(x)dx$$

Also, expectation of function g of X is

$$E(g(x))=\int_{-\infty}^{+\infty} g(x)f(x)\,dx$$

The variance $Var(x)$ or σ^2 is denoted and defined by

$$\sigma^2=Var(X)=E(X^2)-[E(x)]^2$$

The standard deviation is

$$\sigma=SD=\sqrt{Var(X)}$$

Properties of $E(X)$ and $Var(X)$

$\forall a,b\in\mathbb{R}$

1. $E(a)=a$
2. $E(aX)=aE(X)$
3. $E(aX+b)=aE(X)+b$
4. $E(X+Y)=E(X)+E(Y)$
5. $E(aX^2+b)\neq aE(X^2)+b$
6. $\operatorname{var}(a)=0$
7. $\operatorname{var}(aX)=a^2\operatorname{var}(X)$
8. $\operatorname{var}(aX+b)=a^2\operatorname{var}(X)$

End of unit assessment

1. A problem of statistics is given to three students A, B and C whose chances of solving it are $\frac{1}{2}, \frac{3}{4}$ and $\frac{1}{4}$ respectively. What is the probability that the problem will be solved?

2. A company owns 400 laptops. Each laptop has an 8% probability of not working. Suppose you randomly selected 20 laptops for your sales people. What is the likelihood that:

   a) 5 will be broken? b) They will all work?

   c) They will all be broken?

3. A study indicates that 4% of American teenagers have tattoos. If a random sample of 30 teenagers was made, what is the likelihood that exactly 3 will have a tattoo?

4. An XYZ cell phone is made from 55 components. Each component has a 0.002 probability of being defective. What is the probability that an XYZ cell phone will not work perfectly?

5. The ABC Company manufactures toy robots. About 1 toy robot per 100 does not work. Suppose you purchase 35 ABC toy robots, what is the probability that exactly 4 do not work?

6. A thin but biased coin has a probability of 0.55 of landing with the head up and 0.45 of landing with the tail up. The coin is tossed three times. (Determine all numerical answers to the following questions to 6 decimal places).

   a) What is the sample space of possible outcomes of the three tosses?

   b) What is the probability of each of these possible outcomes?

   c) Find the probability function for the number X of the times heads come up during the 3 tosses.

   d) What is the probability that the number of heads is at least one?

   e) What is the expected outcomes of X?

7. Assuming half the population of a town consumes chocolates and that 100 investigators each take 10 individuals to see whether they are consumers, how many investigators would you expect to report that three people or less were consumers?

8. The mean and variance of binomial distribution are 4 and $\frac{4}{3}$ respectively. Find $P(X \geq 1)$.

9. The incidence of occupational disease in an industry is such that the workers have a 25% chance of suffering from it. What is the probability that out of six workers 4 or more will contact the disease?

10. A box contains 'a' red and 'b' black balls, 'n' balls are drawn. Find the expected number of red balls drawn.

11. The probability of a student arriving at the school late on any given day is $\frac{1}{10}$.

    a) What is the probability of his/her being punctual for a whole week (i.e. 5 days).

    b) Calculate the mean and variance of the number of days he/she will be late in school term consisting 14 weeks.

12. Assuming that, on average, one telephone number out of 15 calling between 2 p.m. and 3 p.m. on week day is busy. What is the probability that if 6 randomly selected telephone numbers are called:

    a) Not more than three will be busy?

    b) At least three of them will be busy ?

13. A candidate is selected for interview for three posts. For the first post, there are three candidates, for the second there are four, and for the third one are two. What is the chance of getting at least one post?

14. A cross word puzzle is published in the times magazine each day of the week, except Sunday. A man is able to complete on average 8 out of 10 of the cross puzzles.

    a) Find the expected value and the standard deviation of the number of completed cross words in a given week.

    b) Find the probability that he will complete at least 5 in a given week.

15. The number of accidents in a year involving taxi drivers in a city follows distribution with mean equal to 3. Out of 1000 taxi drivers sampled (selected), find approximately the number of drivers with:

    a) No accident in year.

    b) More than 3 accidents in a year.

16. In a Poisson distribution $P(x)$ for $x = 0$ is 0.1. Find the mean.

17. Six coins are tossed 6400 times. Using the Poisson distribution, what is the approximate probability of getting six heads x times?

18. Suppose that a book of 600 pages contains 40 printing mistakes. Assume that these errors are randomly distributed throughout the book and x, the number of errors per page follows a Poisson distribution. What is the probability that 10 pages selected at random will be free of errors?

19. If I receive 4 e-mails per day via my home computer, what is the probability that on a given day, I receive:

   a) Exactly two e-mails,

   b) No e-mail,

   c) At least three e-mails.

20. Telephone calls arriving at the school office follow a poisson distribution with an average rate of 0.6 per minute. Determine the probability that:

   a) The office receives at least 2 calls in a given minute.

   b) The office receives 7 calls in a space of 10 minutes.

   c) The office receives only 3 calls in a given 5 minutes.

   d) No call arrived while the secretary was out of the office for 6 minutes.

21. Fit a Poisson's distribution to the following data and calculate expected frequencies:

Deaths	0	1	2	3	4
Frequencies	122	160	15	2	1

22. An insurance company found that only 0.01% of the population is involved in a certain type of accident each year. If its 1000 policy holders were randomly selected from the population, what is the probability that not more than two of its clients are involved in such an accident next year? $\left\{e^{-0.1} = 0.9048\right\}$.

23. A manufacturer of coffer pins knows that 5 per cent of his product is defective. If he sells coffer pins in boxes of 100 and guarantees that not more than 4 pins will be defective, what is the approximate probability that a box will fail to meet the guaranteed quality? ($e^{-5} = 0.0067$).

24. Fit a Poisson distribution to the following data which gives the number of yeast cells per square for 400 squares

Mistakes per day	0	1	2	3	4	5	6
Number of days	143	90	42	12	9	3	1

It is given that $e^{-1.32} = 0.2674$

25. A probability distribution function is given by

$$f(x)=\begin{cases}\dfrac{x}{6}, & 0\le x\le 3\\ \dfrac{1}{2}(4-x), & 0\le x\le 3\\ 0, & elsewhere\end{cases}$$

Calculate:

a) The probability that x lies in the interval $(1,2)$.

b) The probability that $x>2$.

26. The probability density function of a continuous random variable X is

$$f(x)=\begin{cases}k\sin x, & 0\le x\le \pi,\\ 0, & elsewhere.\end{cases}$$

Find;

a) The value of k b) $P\left(x>\dfrac{\pi}{3}\right)$

where k is a constant.

27. The time taken to perform a particular task, t hours, has the p.d.f given by

$$f(x)=\begin{cases}10Ct^2, & 0<t<0.6\\ 9C(1-t), & 0.6<t<1\\ 0, & elsewhere\end{cases}$$

where c is a constant.

Determine the probability that the time required will be:

a) More than 48 minutes.

b) Between 24 and 48 minutes.

28. A continuous random variable X has probability density function

$$f(x)=\begin{cases}kx(-x), & 0\le x\le 2,\\ \mathrm{k}(4-x), & 2\le x\le 4,\\ 0, & otherwise.\end{cases}$$

where k is a constant.

Determine the:

a) Value of k b) Mean c) $P(1 \le x \le 3)$

29. The continuous random variable X has probability density function

$$f(x)=\begin{cases} k, & 0 \le x \le 2, \\ k(3-x), & 2 \le x \le 3, \\ 0, & elsewhere. \end{cases}$$

where k is a constant.

a) Determine the:

(i) Value of k

(ii) Mean

(iii) Standard deviation (σ) of X

b) if $E(x)=\mu$, find the $p(x<\mu-\sigma)$

30. For each of the following functions, find the:

a) Value of k for which f is a probability density on the given interval.

b) Mean μ, variance σ^2 and standard deviation σ of the probability density function f, and $p(\mu-\sigma \le x \le \mu+\sigma)$

(i) $f(x)=kx$ on $[0,3]$

(ii) $f(x)=kx^2$ on $[0,1]$

(iii) $f(x)=k(x-x^2)$ on $[0,1]$

31. Petrol is delivered to a garage every Monday morning. At this garage, the weekly demand of petrol in thousands of units is continuous random variable X distributed with a p.d.f of the form

$$f(x)=\begin{cases} ax^2(b-x), 0 \le x \le 1 \\ 0, & otherwise \end{cases}$$

a) Given that the mean weekly demand is 600 units, determine the value of a and b.

b) If the storage tanks at this garage are filled to their capacity of 900 units every Monday morning, what is the probability that in any given week, the garage will be unable to meet the demand of petrol?

Summative Evaluations

The following evaluations cover contents of Senior Four, Senior Five and Senior Six.

For each evaluation there are two sections: A and B

- SECTION A: Attempt ALL questions (**55 marks**)
- SECTION B: Attempt any THREE questions (**45 marks**)

Evaluation 1

SECTION A: Attempt all questions (55 marks)

1. Solve: $\left(\frac{1}{8}\right)^{x-2} = 4^{3-2x}$ **(3 marks)**

2. For which values of m does the following quadratic equation; $x^2+3x+m=0$ admit a double root? Find that root. **(4 marks)**

3. Find the value of k if the angle between $\vec{u}=(k,3)$ and $\vec{v}=(4,0)$ is 45^o. **(2 marks)**

4. Solve: $2\cos^2 x-\cos x-1=0$ **(4 marks)**

5. Evaluate: $\lim\limits_{x\to 0}\frac{1-\cos x}{\sin x}$ **(2 marks)**

6. Consider a sequence $\{u_n\}$ where $u_{n+1}=3(u_n+2)$ and $u_0=0$. List the first five terms of this sequence. Is the sequence arithmetic, geometric or neither? **(4 marks)**

7. Let $f(x)=\sqrt{x^2+2\sqrt{x^2-1}}-\sqrt{x^2-2\sqrt{x^2-1}}$

    a) Find domain of definition of $f(x)$. **(3marks)**

    b) Simplify $f(x)$ on its domain of definition

(**Hint**: Start by calculating $f^2(x)$) **(4 marks)**

8. A biased coin is such that head is three times as likely to appear as tail.
   Find $P(T)$ and $P(H)$. **(4 marks)**

9. Find equations of the tangent and normal lines to the curve of the function $y = f(x) = x^3 - 2x^2 + 4$ at point $(2,4)$. **(3 marks)**

10. a) Give the equation of sphere with centre $(6,5,-2)$ and radius $\sqrt{70}$. **(2 marks)**

    b) Find the radius and the centre of the sphere whose equation is $x^2 + y^2 + z^2 + 4x - 8y + 6z + 7 = 0$
    (3 marks)

    c) Find the intersection of the given sphere in b) and the line passing through the points $A(1,1,-1)$ and $B(2,-3,4)$. **(3 marks)**

11. Sugar dissolves in water at a rate proportional to the amount still undissolved. If there were 50 kg of sugar present initially, and at the end of 5 hours only 20 kg of sugar is left, how much longer will it take until
    90% of the sugar is dissolved? **(4 marks)**

12. Prove that: $\sin y \cos(x-y) + \cos y \sin(x-y) = \sin x$
    (2 marks)

13. Evaluate the following limit:
    $$\lim_{x \to 0} \frac{\sqrt[5]{(1+x)^3 - 1}}{x}$$ **(3 marks)**

14. Find the value of x if the mean of $56, 37, 54, 52, x$ and 48 is 50. **(2 marks)**

15. Calculate the area enclosed by the curves $y^2 = 2px$ and $x^2 = 2py$ **(4 marks)**

SECTION B: Attempt any three questions (45marks)

16. Given the statistical distribution

x_i	7	8	9	11	15
y_i	33	25	17	9	6

a) Calculate the linear correlation coefficient. **(6 marks)**

b) Determine the equation of the regression line of y on x. **(4 marks)**

c) Draw a scatter plot of this set of the distribution and the regression line. **(5 marks)**

17. a) Solve in $\mathbb{R}^3$ the following system

$$\begin{cases} 3x + 2y - 5z = 2 \\ x + 2y \qquad = 3 \\ 2x - y + z = -3 \end{cases}$$ **(8 marks)**

b) Find the area of a parallelogram having adjacent sides $\vec{a} = 6\vec{i} + 3\vec{j} - 2\vec{k}$ and $\vec{b} = 3\vec{i} - 2\vec{j} + 6\vec{k}$ **(7 marks)**

18. A random variable *X* has probability density function

$$F(x) = \begin{cases} cx(6-x)^2 \,;\ 0 \le x \le 6 \\ 0, \qquad\qquad elsewhere \end{cases}$$

a) Find the value of *c* **(4 marks)**

b) Calculate the;

(i) mean **(5 marks)**

(ii) variance **(4 marks)**

(iii) standard deviation **(2 marks)**

19. Given that $I_n = \int_0^{\frac{\pi}{2}} e^{-nx} \sin x dx$ and $J_n = \int_0^{\frac{\pi}{2}} e^{-nx} \cos x dx$, $n \in \mathbb{N}_o$

a) Applying successive integration by parts on integrals I_n and J_n, establish two relations between I_n and J_n.**(9 marks)**

b) Hence, deduce the value of I_n and J_n. **(6 marks)**

20. Solve the differential equation $y''-y'-2y=6x$ given that $y(0)=y'(0)=1$ **(15 marks)**

Evaluation 2

SECTION A: Attempt all questions (55 marks)

1. Solve the following simultaneous equations: $\begin{cases} x^2+y^2=\dfrac{37}{4} \\ xy=\dfrac{3}{2} \end{cases}$ **(2 marks)**

2. Assume that *x* is a positive real number, calculate: $\ln\left(\dfrac{e^{\ln x}}{e^3}\right)=\ln\left(\dfrac{e}{x}\right)$ **(2 marks)**

3. Solve the following equation in $\mathbb{R}$: $\arctan x+\arctan\sqrt{3}=\dfrac{\pi}{4}$ **(3 marks)**

4. Calculate the derivative of the function: $f(x)=\dfrac{\ln\left(1+x^2\right)}{e^{x^2}}$ **(4 marks)**

5. Evaluate the following limit: $\lim\limits_{x\to 1}\dfrac{\sqrt{x}-1}{\sqrt[3]{x}-1}$ **(4 marks)**

6. Solve in the set of complex numbers, the equation $iz-2=4i-z$ and put the answer in algebraic form. **(4 marks)**

7. Prove that $\dfrac{\sin x+\sin 2x}{1+\cos x+\cos 2x}=\tan x$ **(4 marks)**

8. Evaluate the following limit:
$\lim\limits_{x\to+\infty}\dfrac{1+2+3+\ldots+n}{n^2}$ **(4 marks)**

9. Find the equation of the tangent to the curve $y = \ln(4x - 11)$ at the point where $x = 3$. **(5 marks)**

10. The function *f* is defined as follows: $f : IR \to IR : x \to f(x) = \ln\left(\dfrac{x+1}{x-1}\right)$.

    Find the domain of definition of f. **(5 marks)**

11. Given the function $F(x) = \dfrac{x^2}{2} + x - x\ln x$,

    calculate its derivative $F'(x)$ **(5marks)**

12. Solve in $\mathbb{R}$:

    a) $e^x e^{x-1} = e$ **(2 marks)**

    b) $e^{2x-2} + e^{x-2} = 6e^{-2}$ **(2 marks)**

13. Determine the inverse f^{-1} of the function $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ **(3 marks)**

14. Express y in terms of x given that:
    $5\log_2 y - 3\log_2(x+4) = 2\log_2 y + 3\log_2 x$ **(3 marks)**

15. A point P is 90 m away from a vertical flag pole which is 11 m high. What is the angle of elevation to the top of the flag pole from P? **(3 marks)**

SECTION B: Attempt any three questions (45 marks)

16. Solve the equation in the complex number set and the system in $\mathbb{R}^2$.

    a) $z^4 - (8i - 1)z^2 - 8i = 0$ **(11 marks)**

    b) $\begin{cases} 1 + \log_2(-x + 2y) = \log_2(2x - 3y) \\ 3^{5x+y} = \dfrac{81}{3^{-x-7y}} \end{cases}$ **(4 marks)**

17. Given the function *f* of real variable x defined by: $f(x) = \dfrac{x^2 - 1}{x^2 - 4}$

    a) Determine the domain of definition of $f(x)$. **(2 marks)**

    b) Calculate the limits at the boundaries of the domain. **(3 marks)**

c) State any asymptotes. **(2 marks)**

d) Make the variation table. **(3 marks)**

e) Find the *x*-intercepts and y-intercepts for the graph of *f.* **(2 marks)**

f) Sketch the graph of *f* in a Cartesian plane. **(3 marks)**

18. a) From a group of 4 men and 5 women, how many committees of size 3 are possible.

(i) With no restrictions? **(3 marks)**

(ii) With 1 man and 2 women? **(3 marks)**

(iii) With 2 men and 1 woman if a at least a man must be in the committee? **(3 marks)**

b) If 3 books are picked at random from a shelf containing 5 novels, 3 books of poems, and a dictionary. What is the probability that:

(i) The dictionary is picked? **(3 marks)**

(ii) 2 novels and 1 book of poems are selected? **(3 marks)**

19. a) The numerical function *f* of real variable *x* is defined as follows: $f(x)=\dfrac{x^2+x+2}{x+1}$. By writing

$$f(x)=ax+b+\frac{c}{x+1}$$

where a, b and c are real numbers, determine the values of a, b and c; and hence deduce $\int f(x)\,dx$. **(8 marks)**

b) Solve the differential equation

$$\frac{2y}{x}\frac{dy}{dx}=\frac{y^2}{x^2}-1$$

(7 marks)

20. A school office receives 5 calls on average between 09:00 hrs and 10:00 hrs on each weekday. Find the probability that the office:

a) Receives 6 calls between 09:00 hrs and 10:00 hrs on this Wednesday. **(5 marks)**

b) Will receive exactly 3 calls between 09:15 hrs and 09:30 hrs. **(5 marks)**

c) Will receive 3 calls between 09:15 hrs and 09:30 hrs on exactly 2 days during a given week. **(5 marks)**

Evaluation 3

SECTION A: Attempt all questions (55 marks)

1. Solve the following inequality: $3-5x-x^2 \geq 0$ **(3 marks)**

2. Find the equation of the circle passing through points $(0,1),(4,3)$ and $(1,-1)$ **(3 marks)**

3. Determine the value(s) of *k* for which the equation $\dfrac{x^2-x+1}{x-1}=k$ has repeated roots. **(4 marks)**

4. Solve the following system by Gaussian elimination method
$$\begin{cases} x+y-z=-1 \\ 3x-2y+z=0 \\ 2x+3y-3z=-3 \end{cases}$$ **(4 marks)**

5. Evaluate the following limit: $\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2+2}}{3x-6}$ **(3 marks)**

6. An arithmetic series has $U_n = 72-6n$.
If the sum of the first *n* terms of the series is 378; find *n*. **(4 marks)**

7. Find the centre and radius of the sphere with equation: $(x-1)(x-2)+(y+3)(y-4)+(z+1)(z-1)=0$ **(3 marks)**

8. Using a diagrams show the validity or fallacy of the following arguments:
    a) All human being are mortal.
    b) Peter the cat is mortal.

c) Therefore, Peter the cat is a human being. **(4 marks)**

9. Find the derivative of function *f* defined by $f(x)=\sin^2 x\tan x$.**(4 marks)**

10. a) Find the equation of the line joining $A(3,4,1)$ and $B(5,1,6)$. **(2 marks)**

    b) Find the co-ordinates of the point where that line cuts the plane $z=0$. **(2 marks)**

11. Determine the Maclaurin series of the function $f(x)=\cos 3x$. **(3 marks)**

12. Express $\dfrac{x^2+1}{x^3+4x^2+3x}$ in partial fractions **(2.5 marks)**

    and hence $\displaystyle\int\frac{x^2+1}{x^3+4x^2+3x}dx$ **(2.5 marks)**

13. Express the complex number $z=\dfrac{2-2i}{1+i}$ in both algebraic and polar forms. **(3 marks)**

14. Given that $I=\displaystyle\int_0^{\ln 16}\frac{e^x+3}{e^x+4}dx$ and $J=\displaystyle\int_0^{\ln 16}\frac{dx}{e^x+4}$,

    calculate the values of $I+J$ and $I-3J$. **(5 marks)**

15. Let *U* and *W* be the following subspaces of $\mathbb{R}^4$:
    $U=\{(a,b,c,d):b+c+d=0\}$, $W=\{(a,b,c,d):a+b=0,c=2d\}$. Find the dimension of $U\cap W$. **(3 marks)**

SECTION B: Attempt any three questions (45 marks)

16. Suppose you have $Frw100,000$ to invest for one year at a nominal annual rate of interest of 8%, how much would your investment be worth after one year if interest is compounded:

    a) Annually **(3 marks)**

    b) Quarterly **(3 marks)**

c) Monthly **(3 marks)**

d) Weekly **(3 marks)**

e) Daily? **(3 marks)**

17. The following table gives a number of advertisement (x_i) and the volume of sales in hundreds of dollars (y_i) of a certain sports company.

x_i	1	2	3	4	5	6
y_i	41	50	54	54	57	63

a) Find the standard deviation for x_i and y_i. **(4 marks)**

b) Calculate the correlation coefficient r. **(3 marks)**

c) Find the equation of regression line for *y* with respect to x. **(4 marks)**

d) For 7 numbers of advertisements, estimate the volume of sales. **(2 marks)**

18. The vertices of the triangle are $A(1,2,3)$, $B(-2,1,-4)$ and $C(3,4,-2)$.

a) Find the perimeter of the triangle ABC. **(4 marks)**

b) Determine the coordinates of centre of gravity of the triangle ABC. **(3 marks)**

c) Find the angles of the triangle ABC. **(6 marks)**

d) Find the area of the triangle ABC. **(2 marks)**

19. Given the function *f* of real variable x defined by

$$f(x) = x + |x| + 1 - \frac{1}{x+2}$$

a) What is the domain of definition of $f(x)$? **(1 mark)**

b) Write $f(x)$ without the symbol of absolute value. **(2 marks)**

c) Calculate the limit on boundaries of

domain of definition and deduce equation of asymptotes. **(3 marks)**

d) Compute the first derivative and indicate the interval of increasing or decreasing. **(2 marks)**

e) Construct the table of variation. **(2 marks)**

f) Establish the direction of concavity. **(2 marks)**

g) Plot the curve in Cartesian plane. **(3 marks)**

20. a) Find the equation of parabola whose focus is at $(-1,-2)$ and directrix $x-2y+3=0$. **(4 marks)**

b) Find the equation of the set of the all points whose distances from $(0,4)$ are $\frac{2}{3}$ of their distances from the line $y=9$. **(5 marks)**

c) In the hyperbola $x^2-4y^2=4$, find the axes , the coordinates of the foci, the eccentricity and the latus rectum. **(6 marks)**

Evaluation 4

SECTION A: Attempt all questions (55 marks)

1. Find the term independent of x in the expansion of $\left(x+\frac{1}{x}\right)^{20}$.**(3 marks)**

2. The cubic polynomial $6x^3+7x^2+ax+b$ has a remainder of 72 when divided by $(x-2)$ and is exactly divisible by $(x+1)$. Find the values of a and b. **(3 marks)**

3. Solve the equation: $\sin x+\sqrt{3}\cos x=1$ **(3 marks)**

4. Given the matrix $A=\begin{pmatrix}11-x & 2 & 8\\ 2 & 2-x & -10\\ 8 & -10 & 5-x\end{pmatrix}$,

   find the possible values of x such that matrix A is singular (has no inverse) if 9 is one of those values. (3 marks)

5. Solve the following system
   $\begin{cases}\log(x+y)=1\\ \log_2 x+2\log_4 y=4\end{cases}$ **(3 marks)**

6. If α and β are the roots of the equation $x^2-x-3=0$, without solving the equation, find the value of $\alpha^3+\beta^3$. **(3 marks)**

7. Find the centre, foci, and eccentricity for the ellipse:
   $x^2+4y^2-4x+8y+4=0$ **(4 marks)**

8. Write the equation of the tangent and the normal to the curve of $3x^2-xy-2y^2+12=0$ at the point $(2,3)$. **(4 marks)**

9. Find the value of the constant k
   if $\int_0^1 \frac{1}{(2x+k)^2}dx=\frac{1}{3}$ **(5 marks)**

10. Solve in set of complex numbers the equation: $z^6=1$ **(5 marks)**

11. The sum of the first six terms of an arithmetic progression is 72 and the second term is seven
    times the fifth term. Find the first term and the common difference. **(5 marks)**

12. Evaluate $\lim_{x\to\frac{\pi}{2}}(\tan x)^{\cos x}$ **(4 marks)**

13. Given that $\overline{x}=6.2,\ \sigma_x=3.03315,\ \overline{y}=2.04,\ \sigma_y=0.461519$ and $r_{xy}=0.957241$, find the regression line of y on x where; $\overline{x},\ \sigma_x,\ \overline{y},\ \sigma_y$ and r_{xy} stand for the mean of x, the standard deviation of x, the mean of y, the standard deviation of y and the correlation coefficient respectively. **(5 marks)**

14. From a pack of 52 cards, two cards are drawn together at random. What is the probability of both cards being kings? **(2 marks)**

15. Find the cosine of the angle and the angle itself (in radians and degrees) between vectors (2, 5) and (−1, 3). **(3 marks)**

SECTION B: Attempt any three questions (45 marks)

16. On the same graph, sketch the curves of functions $y=x^2-5x+4$ and $y=-2x^2+5x+1$. Hence, find the area of the region enclosed between the two curves. **(15 marks)**

17. a) The events A, B and C in the same sample space are such that A and C are mutually exclusive events while A and B are independent events. Given that:

$$P(A)=\frac{2}{3},\ P(C)=\frac{1}{5},\ P(A\cup B)=\frac{4}{5} \text{ and } P(B\cup C)=\frac{13}{25}.$$

(i) Find $P(A\cup C),\ P(B)$ and $P(A\cap B)$ **(6 marks)**

(ii) Are B and C independent events? Justify your answer. **(2 marks)**

b) A hospital diagnoses that a patient has contracted a virus X, but it is known that one could have been from one of the three trains of the virus $X_1,\ X_2$ or X_3. For the patient having virus X, the probability of it being $X_1,\ X_2$ or X_3 is $\frac{1}{2},\frac{3}{8}$ or $\frac{1}{8}$ respectively and the corresponding probabilities of recovery is $\frac{1}{2},\frac{3}{8}$ and $\frac{1}{8}$. Find the probability that if the selected patient recovers, he had virus X_3. **(7 marks)**

18. a) Given the points $A(2,-3,-1), B(3,-4,2)$ and $C(4,-5,2)$, find:

(i) $\overrightarrow{AB} \times \overrightarrow{AC}$ **(3 marks)**

(ii) The area of the triangle ABC. **(2 marks)**

b) The points A and B have coordinates $(2,1,1)$ and $(0,5,3)$ respectively.

(i) Find the equation of the line AB in terms of parameter. **(3 marks)**

(ii) If C is the point with coordinates $(5,-4,2)$, find the coordinates of point D on AB such that CD is perpendicular to AB. **(5 marks)**

(iii) Find the equation of the plane π containing the line AB and parallel to CD. **(2 marks)**

19. Given the complex number $U = \text{cis}\dfrac{2\pi}{5}$

a) Prove that $\dfrac{1}{2}\left(U + \dfrac{1}{U}\right) = \cos\dfrac{2\pi}{5}$. **(3 marks)**

b) Calculate U^5. **(3 marks)**

c) Deduce that $U^4 + U^3 + U^2 + U + 1 = 0$. **(3 marks)**

d) By taking $x = U + \dfrac{1}{U}$, write the real part of the expression in c) in terms of x. **(3 marks)**

e) Solve, in set of real numbers, the expression obtained in d) and deduce the exact value of $\cos\dfrac{2\pi}{5}$. **(3 marks)**

20. a) In how many ways can 5 men, 4 women and 3 children be arranged in a row so that all men, women and the children each sit together?

(3 marks)

b) Solve the following system

$$\begin{cases} {}^{x}C_{y} = {}^{x}C_{y+1} \\ 4\left({}^{x}C_{y}\right) = 5\left({}^{x}C_{y-1}\right) \end{cases}$$

(6 marks)

c) Prove that

$${}^{n-2}C_{m} + 2\left({}^{n-2}C_{m-1}\right) + {}^{n-2}C_{m-2} = {}^{n}C_{m}$$

(6 marks)

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